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A Step by Step Guide on Derivation and Analysis of a New Numerical Method for Solving Fourth-order Ordinary Differential Equations

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Abstract: This manuscript presents a step by step guide on derivation and analysis of a new numerical method to solve initial value problem of fourth order ordinary differential equations. The method adopted hybrid techniques using power series as the basic function. Collocation of the fourth derivatives was done at both grid and off-grid points. The interpolation of the approximate function is also taken at the first four points. The complete derivation of the new technique is introduced and shown here, as well as the full analysis of the method. The discrete schemes and its first, second, and third derivatives were combined together and solved simultaneously to obtain the required 32 family of block integrators. The block integrators are then applied to solve problem. The method was tested on a linear system of equations of fourth order ordinary differential equation in order to check the practicability and reliability of the proposed method. The results are displaced in tables; it converges faster and uses smaller time for its computations. The basic properties of the method were examined, the method has order of accuracy p=10, the method is zero stable, consistence, convergence and absolutely stable. In future study, we will investigate the feasibility, convergence, and accuracy of the method by on some standard complex boundary value problems of fourth order ordinary differential equations. The extension of this new numerical method will be illustrated and comparison will also be made with some existing methods.

Keywords: Derivation, Analysis, Fourth-order Ordinary Differential Equations, New numerical Method, Hybrid Techniques, Convergence, Zero Stability, Consistency, Taylor Series, Order 10, Integrators

1. Introduction

This work proposes the derivation and analysis of a new numerical method for direct approximation of the differential equations of the form.

$$y'''' = f(x, y, y', y'', y'''), y(x_0) = \tau_0, y'(x_0) = \tau_1, y''(x_0) = \tau_2, y'''(x_0) = \tau_3 \quad (1)$$

where $R \times R^m \times R^m \to R^m$ and $y, y_0, y', y'', y''' \in \Re$ are given real constants.

Several authors have devoted a lot of attention to the development of various methods for solving (1) directly without reducing it to system of first order. Numerous numerical methods based on the use of different polynomial functions and non polynomials have been adopted including Power series [1], Legendre and Chebyshev [2], Taylor series [3], power series and exponential functions [4], Legendre polynomial [5], Adomian decomposition [6], Bernstein Polynomial Basis [8], Lucas Polynomial [9] and, Finite-Difference [12, 14].

In this paper, we are motivated by the work of [1] to develop an hybrid block method with four-step and four-hybrid points with characteristic of high order for solving directly fourth order ordinary differential equations. According to [1], power series has great properties, easy to use and gives a better performance. This work is motivated by the need to develop a new numerical method which can handle system of equations of initial value problems of fourth order ordinary differential equations. The new method is expected to converge faster with lesser time of computation. Hence address setback associated with other methods in the literature. The method has order of accuracy P=10, the accessibly and accuracy of the methods shall be illustrated on the table.

2. Description of the Method

The section deals with the step by step derivation of the new numerical method for solution of initial value problems of fourth order ordinary differential equations.

2.1. Derivation of the Method

The approximate solution of the form

$$y(x)=\sum_{j=0}^{12} a_j x^j \tag{2}$$

is considered for numerical approximation of (1). Where a_j are parameters to be determined and k=4 See [1-4] for more details. The first, second, third and fourth derivatives of (2) are

$$y^{'}(x)=\sum_{j=0}^{12} j a_j x^{j-1} \tag{3}$$

$$y^{''}(x)=\sum_{j=0}^{12} j(j-1)a_j x^{j-2} \tag{4}$$

$$y^{'''}(x)=\sum_{j=0}^{12} j(j-2)(j-1)a_j x^{j-3} \tag{5}$$

$$y^{iv}(x)=\sum_{j=0}^{12} j(j-3)(j-2)(j-1)a_j x^{j-4} \tag{6}$$

Substituting (6) into (2) to obtain

$$\sum_{j=0}^{12} j(j-3)(j-2)(j-1)a_j x^{j-4}=f(x,y,y^{'},y^{''},y^{'''}) \tag{7}$$

where $C=$ Points of collocation, $I=$ Points of interpolation and $E=$ Point of evaluation. $C+I-1=12$ Collocate equation (6) at $x=x_{n+i}, i=0,\frac{1}{2},1,\frac{3}{2},2,\frac{5}{2},3,\frac{7}{2},4$, Interpolate (2) at $x=x_{n+i}, i=0,\frac{1}{2},1,\frac{3}{2}$ and evaluating at the end point $x=x_{n+i}, i=4$. This can be represented in matrix as follows,

$$\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 & x_n^{10} & x_n^{11} & x_n^{12}\\
1 & x_{n+\frac{1}{2}} & x_{n+\frac{1}{2}}^2 & x_{n+\frac{1}{2}}^3 & x_{n+\frac{1}{2}}^4 & x_{n+\frac{1}{2}}^5 & x_{n+\frac{1}{2}}^6 & x_{n+\frac{1}{2}}^7 & x_{n+\frac{1}{2}}^8 & x_{n+\frac{1}{2}}^9 & x_{n+\frac{1}{2}}^{10} & x_{n+\frac{1}{2}}^{11} & x_{n+\frac{1}{2}}^{12}\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 & x_{n+1}^{10} & x_{n+1}^{11} & x_{n+1}^{12}\\
1 & x_{n+\frac{3}{2}} & x_{n+\frac{3}{2}}^2 & x_{n+\frac{3}{2}}^3 & x_{n+\frac{3}{2}}^4 & x_{n+\frac{3}{2}}^5 & x_{n+\frac{3}{2}}^6 & x_{n+\frac{3}{2}}^7 & x_{n+\frac{3}{2}}^8 & x_{n+\frac{3}{2}}^9 & x_{n+\frac{3}{2}}^{10} & x_{n+\frac{3}{2}}^{11} & x_{n+\frac{3}{2}}^{12}\\
0 & 0 & 0 & 0 & 24 & 120x_n & 360x_n^2 & 840x_n^3 & 1680x_n^4 & 3024x_n^5 & 5040x_n^6 & 7920x_n^7 & 11880x_n^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{1}{2}} & 360x_{n+\frac{1}{2}}^2 & 840x_{n+\frac{1}{2}}^3 & 1680x_{n+\frac{1}{2}}^4 & 3024x_{n+\frac{1}{2}}^5 & 5040x_{n+\frac{1}{2}}^6 & 7920x_{n+\frac{1}{2}}^7 & 11880x_{n+\frac{1}{2}}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+1} & 360x_{n+1}^2 & 840x_{n+1}^3 & 1680x_{n+1}^4 & 3024x_{n+1}^5 & 5040x_{n+1}^6 & 7920x_{n+1}^7 & 11880x_{n+1}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{3}{2}} & 360x_{n+\frac{3}{2}}^2 & 840x_{n+\frac{3}{2}}^3 & 1680x_{n+\frac{3}{2}}^4 & 3024x_{n+\frac{3}{2}}^5 & 5040x_{n+\frac{3}{2}}^6 & 7920x_{n+\frac{3}{2}}^7 & 11880x_{n+\frac{3}{2}}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 & 3024x_{n+2}^5 & 5040x_{n+2}^6 & 7920x_{n+2}^7 & 11880x_{n+2}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{5}{2}} & 360x_{n+\frac{5}{2}}^2 & 840x_{n+\frac{5}{2}}^3 & 1680x_{n+\frac{5}{2}}^4 & 3024x_{n+\frac{5}{2}}^5 & 5040x_{n+\frac{5}{2}}^6 & 7920x_{n+\frac{5}{2}}^7 & 11880x_{n+\frac{5}{2}}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+2} & 360x_{n+2}^2 & 840x_{n+2}^3 & 1680x_{n+2}^4 & 3024x_{n+2}^5 & 5040x_{n+2}^6 & 7920x_{n+2}^7 & 11880x_{n+2}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+\frac{7}{2}} & 360x_{n+\frac{7}{2}}^2 & 840x_{n+\frac{7}{2}}^3 & 1680x_{n+\frac{7}{2}}^4 & 3024x_{n+\frac{7}{2}}^5 & 5040x_{n+\frac{7}{2}}^6 & 7920x_{n+\frac{7}{2}}^7 & 11880x_{n+\frac{7}{2}}^8\\
0 & 0 & 0 & 0 & 24 & 120x_{n+4} & 360x_{n+4}^2 & 840x_{n+4}^3 & 1680x_{n+4}^4 & 3024x_{n+4}^5 & 5040x_{n+4}^6 & 7920x_{n+4}^7 & 11880x_{n+4}^8
\end{bmatrix}
\begin{bmatrix} a_0\\ a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\\ a_7\\ a_8\\ a_9\\ a_{10}\\ a_{11}\\ a_{12}\end{bmatrix}
=
\begin{bmatrix} y_n\\ y_{n+\frac{1}{2}}\\ y_{n+1}\\ y_{n+\frac{3}{2}}\\ f_n\\ f_{n+\frac{1}{2}}\\ f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3}\\ f_{n+\frac{7}{2}}\\ f_{n+4}\end{bmatrix} \tag{8}$$

Solve for $a_j, j=0(1)12$ in (8) using Gaussian elimination method with the help of Computer Aided Software gives:

$$y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h^4\left(\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\beta_v(x)f_{n+v}\right) \tag{9}$$

where $y(x)$ is the numerical solution of the initial value problem and $v=\frac{1}{2},\frac{3}{2},\frac{5}{2},\frac{7}{2}$. α_j and β_j are constants.

$f_{n+j}=f(x_{n+j},y_{n+j},y'_{n+j},y''_{n+j},y'''_{n+j})$,

using the transformation $t=\frac{x-x_{n+3}}{h}, \frac{dt}{dx}=\frac{1}{h}$

The coefficients of y_{n+j} and f_{n+j} are obtained as:

$$\alpha_0(t)=\frac{-1}{3}(-3+36t+24t^2+4t^3),\quad \alpha_{\frac{1}{2}}(t)=(36+54t+26t^2+4t^3),$$

$$\alpha_1(t)=(-45-63t-28t^2-4t^3)\ ,\ \alpha_{\frac{3}{2}}(t)=\frac{1}{3}(60+74t+30t^2+4t^3)$$

$$\beta_0(t)=\frac{h^4}{3832012800}\begin{bmatrix}-836550-1171485t-565061t^2-249876t^3+380160t^5+198528-228096t^6\\-228096t^7-209088t^8+50688t^{10}+18432t^{11}+2048t^{12}\end{bmatrix}$$

$$\beta_{\frac{1}{2}}(t)=\frac{-h^4}{479001600}\begin{bmatrix}-49289130-77044707t-39299207t^2-6737984t^3+456192t^5+228096t^6\\-278784t^7-240768t^8+57040t^{10}+19968t^{11}+2048t^{12}\end{bmatrix}$$

$$\beta_1(t)=\frac{h^4}{958003200}\begin{bmatrix}447663150+726110865t+386383357t^2+65955296t^3+3991680t^5+1862784t^6\\-2496384t^7-1970496t^8+140800t^9+489984t^{10}+150528t^{11}+14336t^{12}\end{bmatrix} \quad (10)$$

$$\beta_2(t)=\frac{h^4}{383201280}\begin{bmatrix}40333590+108979461t+101714005t^2+29214988t^3+7983360t^5+2395008t^6\\-5220864t^7-2578752t^8+619520t^9+692736t^{10}+172032t^{11}+14336t^{12}\end{bmatrix}$$

$$\beta_3(t)=\frac{h^4}{958003200}\begin{bmatrix}9386001+5584590t+14988421t^2+33454872t^3+39916800t^4+15168384t^5-9787008t^6\\-10378368t^7-1463616t^8+1774080t^9+963072t^{10}+193536t^{11}+14336t^{12}\end{bmatrix}$$

$$\beta_{\frac{7}{2}}(t)=\frac{h^4}{479001600}\begin{bmatrix}-678150-1012725t-787081t^2-921448t^3+2280960t^5+2963248t^6\\+1343232t^7-139392t^8-387200t^9-160512t^{10}-29184t^{11}-2048t^{12}\end{bmatrix}$$

$$\beta_4(t)=\frac{h^4}{383201280}\begin{bmatrix}589050+873315t+577003t^2+537020t^3-1140480t^5-1140480t^6+\\126720t^7+804672t^8+563200t^9+185856t^{10}+30720t^{11}+2048t^{12}\end{bmatrix}$$

Evaluating (10) at t=1 gives the discrete scheme below

$$y_{n+4}+140y_{n+1}-120y_{n+\frac{1}{2}}-56y_{n+\frac{3}{2}}+35y_n=\frac{h^4}{829440}\begin{bmatrix}-565f_n+298040f_{n+\frac{1}{2}}+1409780f_{n+1}+\\1022792f_{n+\frac{3}{2}}+615410f_{n+2}+192200f_{n+\frac{5}{2}}\\+86420f_{n+3}+4280f_{n+\frac{7}{2}}+443f_{n+4}\end{bmatrix} \quad (11)$$

First derivative of (10) gives

$$\alpha_0'(t)=\frac{-1}{3h}(36+48t+12t^2)\ ,\ \alpha_{\frac{1}{2}}'(t)=\frac{1}{h}(54+52t+12t^2)\ ,\alpha_1'(t)=\frac{1}{h}(-63-84t-12t^2)\ ,\alpha_{\frac{3}{2}}'(t)=\frac{1}{3h}(74+60t+12t^2)$$

$$\beta_0'(t)=\frac{h^3}{3832012800}\begin{bmatrix}-1171485-1130122t-749628t^2+1900800t^4+1191168t^5-\\156672t^6-1672704t^7+506880t^9+202752t^{10}+24576t^{11}\end{bmatrix}$$

$$\beta_{\frac{1}{2}}'(t)=\frac{h^3}{479001600}\begin{bmatrix}-77044707-78598414t-20213952t^2+2280960t^4+1368576t^5-\\1951488t^6-1926144t^7+63360t^8+591360t^9+219648t^{10}+124576t^{11}\end{bmatrix} \quad (12)$$

$$\beta_1'(t)=\frac{h^3}{958003200}\begin{bmatrix}726110865+77276670t+19786588t^2+19958400t^4+11176704t^5-17474688t^6\\-15763968t^7+1267200t^8+4899840t^9+1655808t^{10}+172032t^{11}\end{bmatrix}$$

$$\beta'_{\frac{3}{2}}(t)=\frac{-h^3}{479001600}\begin{bmatrix}-254454165-345068816t-123540120t^2+26611200t^4+13128192t^5-\\23950080t^6-18551808t^7+2851200t^8+5829120t^9+1774080t^{10}+172032t^{11}\end{bmatrix}$$

$$\beta'_2(t)=\frac{h^3}{3832012800}\begin{bmatrix}108979461+203428010t+87644964t^2+39916800t^4+14370048t^5-\\36546048t^6-20630016t^7+5575680t^9+1892352t^{10}+172032t^{11}\end{bmatrix}$$

$$\beta'_{\frac{5}{2}}(t)=\frac{-h^3}{479001600}\begin{bmatrix}-18799965-109771954t-139107936t^2+79833600t^4-3193344t^5-\\55528704t^6-19565568t^7+9820800t^8+8194560t^9-2010624t^{10}+172032t^{11}\end{bmatrix}$$

$$\beta'_3(t)=\frac{h^3}{958003200}\begin{bmatrix}9386001+29976842t+100364616t^2+159667200t^3+75841920t^4-58722048t^5\\-72648576t^6-11708928t^7+15966720t^8+9630720t^9+2128896t^{10}+172032t^{11}\end{bmatrix}$$

$$\beta'_{\frac{7}{2}}(t)=\frac{h^3}{479001600}\begin{bmatrix}-1012725-1574162t-2764344t^2+11404800t^4+17779488t^5+\\9402624t^6-1115136t^7-3484800t^8-1605120t^9-321024t^{10}-24576t^{11}\end{bmatrix}$$

$$\beta'_4(t)=\frac{h^3}{3832012800}\begin{bmatrix}873315+1154006t+1611060t^2-5702400t^4-6614784t^5+\\887040t^6+6437376t^7+5068800t^8+1858560t^9+337920t^{10}+24576t^{11}\end{bmatrix}$$

while the second derivative of (10) gives:

$$\alpha''_0(t)=\frac{-1}{3h^2}(48+24t),\ \alpha''_{\frac{1}{2}}(t)=\frac{1}{h^2}(52+24t),\alpha''_1(t)=\frac{1}{h^2}(-84-24t),\alpha''_{\frac{3}{2}}(t)=\frac{1}{3h^2}(60+24t)$$

$$\beta''_0(t)=\frac{h^2}{3832012800}\begin{bmatrix}-1130122-1499256t+7603200t^3+5955840t^4-940032t^5-\\11708928t^6+4561920t^8+2027520t^9+270336t^{10}\end{bmatrix}$$

$$\beta''_{\frac{1}{2}}(t)=\frac{-h^2}{479001600}\begin{bmatrix}-78598414-40427904t+9123840t^3+6842880t^4-11708928t^5\\-13483008t^6+506880t^7+5322240t^8+2196480t^9+270336t^{10}\end{bmatrix}$$

$$\beta''_1(t)=\frac{h^2}{958003200}\begin{bmatrix}77276670+395731776t+79833600t^3+55883520t^4-104848128t^5\\-110247776t^6+10137600t^7+44098560t^8+16558080t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_{\frac{3}{2}}(t)=\frac{-h^2}{479001600}\begin{bmatrix}-345068816-247080240t+106444800t^3+65640960t^4-143700480t^5\\-129862656t^6+22809600t^7+52462080t^8+17740800t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_{\frac{3}{2}}(t)=\frac{-h^2}{479001600}\begin{bmatrix}-345068816-247080240t+106444800t^3+65640960t^4-143700480t^5\\-129862656t^6+22809600t^7+52462080t^8+17740800t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_2(t)=\frac{h^2}{3832012800}\begin{bmatrix}203428010+175289928t+159667200t^3+71850240t^4-219276288t^5-\\144410112t^6+44605440t^7+62346240t^8+18923520t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_{\frac{5}{2}}(t)=\frac{-h^2}{479001600}\begin{bmatrix}-109771954-278215872t+319334400t^3-15966720t^4-333172224t^5\\-136958976t^6+78566400t^7+73751040t^8-20106240t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_3(t)=\frac{h^2}{958003200}\begin{bmatrix}29976842+200729232t+479001600t^2+303367680t^3-293610240t^4\\-435891456t^5-81962496t^6+127733760t^8+2188960t^9+1892352t^{10}\end{bmatrix}$$

$$\beta''_{\frac{7}{2}}(t)=\frac{-h^2}{479001600}\begin{bmatrix}-1574162-5528688t+45619200t^3+88897440t^4+56415744t^5\\-7805952t^6-27878400t^7-14446080t^8-3210240t^9-270336t^{10}\end{bmatrix}$$

$$\beta''_4(t)=\frac{h^2}{3832012800}\begin{bmatrix}1154006+3222120t-22809600t^3-33073920t^4+5322240t^5\\+45061632t^6+40550400t^8+16727040t^8+3379200t^9+270336t^{10}\end{bmatrix} \quad (13)$$

while the third derivative of (10) gives:

$$\alpha_0'''(t)=\frac{-8}{h^3},\ \alpha_{1/2}'''(t)=\frac{24}{h^3},\ \alpha_1'''(t)=\frac{-24}{h^3},\ \alpha_{3/2}'''(t)=\frac{24}{h^3}$$

$$\begin{aligned}
\beta_0'''(t)&=\frac{h}{3832012800}\begin{bmatrix}-1499256+22809600t^2+23823360t^3-4700160t^4\\ -70253568t^5+36495360t^7+18247680t^8+2703360t^9\end{bmatrix}\\
\beta_{\frac{1}{2}}'''(t)&=\frac{-h}{479001600}\begin{bmatrix}-404277904+27371520t^2+27371520t^3-58544640t^4-80898048t^5\\ +3548160t^6+42577920t^7+19768320t^8+2703360t^9\end{bmatrix}\\
\beta_1'''(t)&=\frac{h}{958003200}\begin{bmatrix}395731776+239500800t^2+223534080t^3-524240640t^4-662086656t^5\\ +70963200t^6+352788480t^7+149022720t^8+18923520t^9\end{bmatrix}\\
\beta_{\frac{3}{2}}'''(t)&=\frac{-h}{479001600}\begin{bmatrix}-247080240+319334400t^2+262563840t^3-718502400t^4-779175936t^5\\ +159667200t^6+419696640t^7+159667200t^8+18923520t^9\end{bmatrix}\\
\beta_2'''(t)&=\frac{h}{3832012800}\begin{bmatrix}175289928+479001600t^2+287400960t^3-1096381440t^4\\ -866460672t^5+312238080t^6+498769920t^7+170311680t^8\end{bmatrix}\\
\beta_{\frac{5}{2}}'''(t)&=\frac{-h}{479001600}\begin{bmatrix}-278215872+958003200t^2-63866880t^3-1665861120t^4-821753856t^5\\ +549964800t^6+408320t^7-180956160t^8+18923520t^9\end{bmatrix}\\
\beta_3'''(t)&=\frac{h}{958003200}\begin{bmatrix}200729232+958003200t+910103040t^2-1174440960t^3-2179457280t^4\\ -4917749t^5+894136320t^6+693411840t^7+19700640t^8+18923520t^9\end{bmatrix}\\
\beta_{\frac{7}{2}}'''(t)&=\frac{h}{479001600}\begin{bmatrix}-5528688+136857600t^2+355589760t^3+282078720t^4-46835712t^5\\ -195148800t^6-115568640t^7-28892160t^8-2703360t^9\end{bmatrix}\\
\beta_4'''(t)&=\frac{h}{3832012800}\begin{bmatrix}3222120-68428800t^2-132295680t^3+26611200t^4+270369792t^5\\ +324403200t^7+133816320t^7+30412800t^8+2703360t^9\end{bmatrix}
\end{aligned} \tag{14}$$

Evaluating (10) at non-interpolate points at $t=0,-1,\frac{-1}{2}$ 2 gives the following discrete schemes

$$y_{n+3}-20y_{n+\frac{3}{2}}+45y_{n+1}-36y_{n+\frac{1}{2}}+10y_n=\frac{-h^4}{3870720}\begin{bmatrix}845f_n-398296f_{n+\frac{1}{2}}-1808740f_{n+1}-\\ 1012840f_{n+\frac{3}{2}}-407410f_{n+2}+15320f_{n+\frac{5}{2}}\\ -22564f_{n+3}+5480f_{n+\frac{7}{2}}-595f_{n+4}\end{bmatrix} \tag{15}$$

$$y_{n+2}-4y_{n+\frac{3}{2}}+6y_{n+1}-4y_{n+\frac{1}{2}}+y_n=\frac{-h^4}{11612160}\begin{bmatrix}329f_n-120280f_{n+\frac{1}{2}}-492004f_{n+1}-\\ 100648f_{n+\frac{3}{2}}-24922f_{n+2}+17624f_{n+\frac{5}{2}}\\ -7492f_{n+3}+1832f_{n+\frac{7}{2}}-199f_{n+4}\end{bmatrix} \tag{16}$$

$$y_{n+\frac{5}{2}}-10y_{n+\frac{3}{2}}+20y_{n+1}-15y_{n+\frac{1}{2}}+4y_n=\frac{-h^4}{11612160}\begin{bmatrix}117f_n-47900f_{n+\frac{1}{2}}-2095460f_{n+1}-\\877880f_{n+\frac{3}{2}}-225410f_{n+2}+70648f_{n+\frac{5}{2}}\\-29060f_{n+3}+700f_{n+\frac{7}{2}}-755f_{n+4}\end{bmatrix} \tag{17}$$

$$y_{n+\frac{7}{2}}-35y_{n+\frac{3}{2}}+84y_{n+1}-70y_{n+\frac{1}{2}}+20y_n=\frac{-h^4}{1658880}\begin{bmatrix}685f_n-341000f_{n+\frac{1}{2}}-1585604f_{n+1}-\\1041320f_{n+\frac{3}{2}}-537410f_{n+2}-92120f_{n+\frac{5}{2}}\\-3626f_{n+3}+4744f_{n+\frac{7}{2}}-515f_{n+4}\end{bmatrix} \tag{18}$$

Evaluating (12) at points $t=-3,\ \frac{-5}{2},-2,\ \frac{-3}{2},\ 0,\ -1\ ,\ \frac{-1}{2}$ 2 gives

$$1277337600hy'_n+4683571200y_n+851558400y_{n+\frac{3}{2}}-3832012800y_{n+1}+7664025600y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-1322633f_n+30018920f_{n+\frac{1}{2}}-6069500f_{n+1}+851558400f_{n+\frac{3}{2}}+6551290f_{n+2}-4396952f_{n+\frac{5}{2}}\\+1873060f_{n+3}-464360f_{n+\frac{7}{2}}+51175f_{n+4}\end{bmatrix} \tag{19}$$

$$3832012800hy'_{n+\frac{1}{2}}+2554675200y_n+1277337600y_{n+\frac{3}{2}}-7664025600y_{n+1}+3832012800y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}81145f_n+21111976f_{n+\frac{1}{2}}+23711020f_{n+1}+1277337600f_{n+\frac{3}{2}}+6467110f_{n+2}-3567560f_{n+\frac{5}{2}}\\+1353484f_{n+3}-310760f_{n+\frac{7}{2}}+32425f_{n+4}\end{bmatrix} \tag{20}$$

$$3832012800hy'_{n+1}-1277337600y_n-2554675200y_{n+\frac{3}{2}}-3832012800y_{n+1}+7664025600y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}68615f_n-13441480f_{n+\frac{1}{2}}-31841164f_{n+1}+2554675200f_{n+\frac{3}{2}}-761910f_{n+2}+4320200f_{n+\frac{5}{2}}\\-1667980f_{n+3}+387704f_{n+\frac{7}{2}}-40825f_{n+4}\end{bmatrix} \tag{21}$$

$$1277337600y'_{n+\frac{3}{2}}+851558400y_n+4683571200y_{n+\frac{3}{2}}+7664025600y_{n+1}+3832012800y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}37735f_n+8965880f_{n+\frac{1}{2}}+31109900f_{n+1}+4683571200f_{n+\frac{3}{2}}+3418310f_{n+2}-2117080f_{n+\frac{5}{2}}\\+86420f_{n+3}-200440f_{n+\frac{7}{2}}+21353f_{n+4}\end{bmatrix} \tag{22}$$

$$1277337600y'_{n+2}+14050713600y_n-33210777600y_{n+\frac{3}{2}}+72808243200y_{n+1}-53648179200y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-334031f_n+144849064f_{n+\frac{1}{2}}+624528604f_{n+1}+33210777600f_{n+\frac{3}{2}}+31957750f_{n+2}-\\ 19141544f_{n+\frac{5}{2}}+8095036f_{n+3}-19983704f_{n+\frac{7}{2}}+216001f_{n+4}\end{bmatrix} \quad (23)$$

$$3832012800y'_{n+\frac{5}{2}}+33210777600y_n-60034867200y_{n+\frac{3}{2}}+145616486400y_{n+1}-118792396800y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-724939f_n+341722760f_{n+\frac{1}{2}}+1559756540f_{n+1}+894237800f_{n+\frac{3}{2}}+308225870f_{n+2}-\\ 45663016f_{n+\frac{5}{2}}+20429660f_{n+3}-4920520f_{n+\frac{7}{2}}+529445f_{n+4}\end{bmatrix} \quad (24)$$

$$425779200y'_{n+3}+6670540800y_n-10502553600y_{n+\frac{3}{2}}+26824089600y_{n+1}-22992076800y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-130165f_n+68484184f_{n+\frac{1}{2}}+322715940f_{n+1}+10502553600f_{n+\frac{3}{2}}+121088290f_{n+2}+\\ 16711080f_{n+\frac{5}{2}}+4171556f_{n+3}-900200f_{n+\frac{7}{2}}+97035f_{n+4}\end{bmatrix} \quad (25)$$

$$3832012800y'_{n+\frac{7}{2}}+94522982400y_n-136675123200y_{n+\frac{3}{2}}+360209203200y_{n+1}-318057062400y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-1804745f_n+970047880f_{n+\frac{1}{2}}+4652709556f_{n+1}+36550359720f_{n+\frac{3}{2}}+2348536090f_{n+2}+\\ 836297560f_{n+\frac{5}{2}}+284743060f_{n+3}-8807816f_{n+\frac{7}{2}}+1377895f_{n+4}\end{bmatrix} \quad (26)$$

$$3832012800hy'_{n+4}+136675123200y_n+186491289600y_{n+\frac{3}{2}}-501993676800y_{n+1}+452177510400y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-2494435f_n+1401489800f_{n+\frac{1}{2}}+6810539180f_{n+1}+5721593336f_{n+\frac{3}{2}}+4117306430f_{n+2}+\\ 1967486840f_{n+\frac{5}{2}}+1040221580f_{n+3}+213576200f_{n+\frac{7}{2}}+5935469f_{n+4}\end{bmatrix} \quad (27)$$

Evaluating (13) at $t=-3,\ \frac{-5}{2},\ -2,\ \frac{-3}{2},\ 0,\ -1,\ \frac{-1}{2}$ 2 gives

$$1916006400h^2y''_n-15328051200y_n+7664025600y_{n+\frac{3}{2}}-30656102400y_{n+1}+3832012800y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}29397487f_n+371214904f_{n+\frac{1}{2}}-40557452f_{n+1}+172563976f_{n+\frac{3}{2}}-164939750f_{n+2}\\ +105890248f_{n+\frac{5}{2}}-44086604f_{n+3}+10778872f_{n+\frac{7}{2}}-1176881f_{n+4}\end{bmatrix} \quad (28)$$

$$1916006400h^2y''_{n+\frac{1}{2}}-7664025600y_n+8127128y_{n+\frac{3}{2}}-7664025600y_{n+1}+15328051200y_{n+\frac{1}{2}}$$
$$=h^4\begin{bmatrix}-959741f_n-39395384f_{n+\frac{1}{2}}+4124548f_{n+1}-8127128f_{n+\frac{3}{2}}+7828450f_{n+2}-5020904f_{n+\frac{5}{2}}\\ +2087524f_{n+3}-509768f_{n+\frac{7}{2}}+55603f_{n+4}\end{bmatrix} \quad (29)$$

$$1916006400h^2 y''_{n+1} + 0y_n - 7664025600y_{n+\frac{3}{2}} + 15328051200y_{n+1} + 1460168y_{n+\frac{1}{2}}$$
$$= \frac{h^4}{123344}\begin{bmatrix} 55603f_n - 1460168f_{n+\frac{1}{2}} - 37393676f_{n+1} - 546104f_{n+\frac{3}{2}} - 1121150f_{n+2} \\ +822472f_{n+\frac{5}{2}} - 350252f_{n+3} + 85816f_{n+\frac{7}{2}} - 9341f_{n+4} \end{bmatrix} \quad (30)$$

$$1916006400h^2 y''_{n+\frac{3}{2}} + 7664025600y_n - 15328051200y_{n+\frac{3}{2}} + 3832012800y_{n+1} - 30656102400y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -226481f_n + 79524472f_{n+\frac{1}{2}} + 322926196f_{n+1} + 29818648f_{n+\frac{3}{2}} + 14725450f_{n+2} - \\ 11576024f_{n+\frac{5}{2}} + 4982548f_{n+3} - 1223096f_{n+\frac{7}{2}} + 133087f_{n+4} \end{bmatrix} \quad (31)$$

$$1916006400h^2 y''_{n+2} + 15328051200y_n - 22992076800y_{n+\frac{3}{2}} + 61312204800y_{n+1} - 53648179200y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -301193f_n + 157345336f_{n+\frac{1}{2}} + 733760884f_{n+1} + 444602248f_{n+\frac{3}{2}} + 79484650f_{n+2} - \\ 25307192f_{n+\frac{5}{2}} + 9492724f_{n+3} - 2226824f_{n+\frac{7}{2}} + 237367f_{n+4} \end{bmatrix} \quad (32)$$

$$1916006400h^2 y''_{n+\frac{5}{2}} + 22992076800y_n + 30656102400y_{n+\frac{3}{2}} + 84304281600y_{n+1} - 76640256000y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -414053f_n + 235716904f_{n+\frac{1}{2}} + 1140058468f_{n+1} + 913105096f_{n+\frac{3}{2}} + 523856050f_{n+2} + \\ 14680888f_{n+\frac{5}{2}} + 9465796f_{n+3} - 2679848f_{n+\frac{7}{2}} + 303499f_{n+4} \end{bmatrix} \quad (33)$$

$$1916006400h^2 y''_{n+3} + 30656102400y_n - 3832012800y_{n+\frac{3}{2}} + 107296358400y_{n+1} - 99632332800y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -565061f_n + 314393656f_{n+\frac{1}{2}} + 1545533428f_{n+1} + 1380275272f_{n+\frac{3}{2}} + 1017140050f_{n+2} + \\ 439087816f_{n+\frac{5}{2}} + 59953684f_{n+3} - 6296648f_{n+\frac{7}{2}} + 577003f_{n+4} \end{bmatrix} \quad (34)$$

$$1916006400h^2 y''_{n+\frac{7}{2}} + 3832012800y_n - 45984153600y_{n+\frac{3}{2}} + 130288435200y_{n+1} - 122624409600y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -508697f_n + 391165912f_{n+\frac{1}{2}} + 1958778964f_{n+1} + 1829203576f_{n+\frac{3}{2}} + 1535220250f_{n+2} + \\ 886278472f_{n+\frac{5}{2}} + 512279668f_{n+3} + 33135976f_{n+\frac{7}{2}} - 446921f_{n+4} \end{bmatrix} \quad (35)$$

$$1916006400h^2 y''_{n+4} + 45984153600y_n - 53648179200y_{n+\frac{3}{2}} + 153280512000y_{n+1} - 145616486400y_{n+\frac{1}{2}}$$
$$= h^4\begin{bmatrix} -1749761f_n + 479822392f_{n+\frac{1}{2}} + 2323412596f_{n+1} + 2394886408f_{n+\frac{3}{2}} + 1871582650f_{n+2} + \\ 1521741256f_{n+\frac{5}{2}} + 878405428f_{n+3} + 521114104f_{n+\frac{7}{2}} + 29901727f_{n+4} \end{bmatrix} \quad (36)$$

Evaluating (14) at $t = -3, \frac{-5}{2}, -2, \frac{-3}{2}, 0, -1, \frac{-1}{2}$ 2 gives

$$14515200h^3y_n''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} -2084463f_n + 38364800f_{n+\frac{1}{2}} + 5902624f_{n+1} - 9225936f_{n+\frac{3}{2}} + 8506010f_{n+2} - \\ 5379392f_{n+\frac{5}{2}} + 22222f_{n+3} - 540784f_{n+\frac{7}{2}} + 58861f_{n+4} \end{bmatrix} \quad (37)$$

$$14515200h^3y_{n+\frac{1}{2}}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} 55571f_n - 1411412f_{n+\frac{1}{2}} - 3306564f_{n+1} + 116121600f_{n+\frac{3}{2}} + 1560230f_{n+2} + 913284f_{n+\frac{5}{2}} \\ -360148f_{n+3} + 84964f_{n+\frac{7}{2}} - 9045f_{n+4} \end{bmatrix} \quad (38)$$

$$14515200h^3y_{n+1}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} -12335f_n + 1339776f_{n+\frac{1}{2}} + 3183008f_{n+1} + 116121600f_{n+\frac{3}{2}} + 1074330f_{n+2} - 596800f_{n+\frac{5}{2}} \\ +228424f_{n+3} - 52848f_{n+\frac{7}{2}} + 5549f_{n+4} \end{bmatrix} \quad (39)$$

$$14515200h^3y_{n+\frac{3}{2}}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} 2259f_n + 348364800f_{n+\frac{1}{2}} + 6459580f_{n+1} + 3723372f_{n+\frac{3}{2}} - 591910f_{n+2} + 198916f_{n+\frac{5}{2}} \\ -55764f_{n+3} + 10340f_{n+\frac{7}{2}} - 917f_{n+4} \end{bmatrix} \quad (40)$$

$$14515200h^3y_{n+2}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} -4207f_n + 1213312f_{n+\frac{1}{2}} + 6027552f_{n+1} + 7543088f_{n+\frac{3}{2}} + 3857050f_{n+2} - 652608f_{n+\frac{5}{2}} \\ +196808f_{n+3}41072f_{n+\frac{7}{2}} + 4077f_{n+4} \end{bmatrix} \quad (41)$$

$$14515200h^3y_{n+\frac{5}{2}}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} 787f_n + 1161900f_{n+\frac{1}{2}} + 6280124f_{n+1} + 6691564f_{n+\frac{3}{2}} + 8306010f_{n+2} + 3167108f_{n+\frac{5}{2}} \\ 235220f_{n+3} + 31716f_{n+\frac{7}{2}} - 2389f_{n+4} \end{bmatrix} \quad (42)$$

$$14515200h^3y_{n+3}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} -5679f_n + 1225088f_{n+\frac{1}{2}} + 5995936f_{n+1} + 7487280f_{n+\frac{3}{2}} + 6639770f_{n+2} + 8430784f_{n+\frac{5}{2}} \\ +3041352f_{n+3} - 167536f_{n+\frac{7}{2}} + 12205f_{n+4} \end{bmatrix} \quad (43)$$

$$14515200h^3y_{n+\frac{7}{2}}''' + 116121600y_n - 116121600y_{n+\frac{3}{2}} + 348364800y_{n+1} - 348364800y_{n+\frac{1}{2}} = h^4\begin{bmatrix} 8915f_n + 1087276f_{n+\frac{1}{2}} + 6584508f_{n+1} + 5977196f_{n+\frac{3}{2}} + 9274330f_{n+2} + 4925700f_{n+\frac{5}{2}} \\ +9530924f_{n+3} + 2583652f_{n+\frac{7}{2}} - 55701f_{n+4} \end{bmatrix} \quad (44)$$

$$\begin{aligned}&14515200h^3 y'''_{n+4}+116121600y_n-116121600y_{n+\frac{3}{2}}+348364800y_{n+1}-348364800y_{n+\frac{1}{2}}\\&=h^4\begin{bmatrix}-58991f_n+1713024f_{n+\frac{1}{2}}+4002080f_{n+1}+12269872f_{n+\frac{3}{2}}-791910f_{n+2}+16116416f_{n+\frac{5}{2}}\\+321736f_{n+3}+11517840f_{n+\frac{7}{2}}+2084333f_{n+4}\end{bmatrix}\end{aligned} \tag{45}$$

2.2. Derivation of the Block Integrators

This is done by combining equations (11), (15), (16), (17), (18), (19), (28) and (37) and solve simultaneously to give block integrators which will be used in the future research to solve complex problems.

$$\begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\\y_{n+\frac{5}{2}}\\y_{n+3}\\y_{n+\frac{7}{2}}\\y_{n+4}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-\frac{1}{2}}\\y_{n-1}\\y_{n-\frac{3}{2}}\\y_{n-2}\\y_{n-\frac{5}{2}}\\y_{n-3}\\y_{n-\frac{7}{2}}\\y_n\end{bmatrix}+$$

$$h^3\begin{bmatrix}0&0&0&0&0&0&0&\frac{1}{48}\\0&0&0&0&0&0&0&\frac{1}{6}\\0&0&0&0&0&0&0&\frac{9}{16}\\0&0&0&0&0&0&0&\frac{4}{3}\\0&0&0&0&0&0&0&\frac{125}{48}\\0&0&0&0&0&0&0&\frac{9}{2}\\0&0&0&0&0&0&0&\frac{343}{48}\\0&0&0&0&0&0&0&\frac{32}{3}\end{bmatrix}\begin{bmatrix}y'''_{n-\frac{1}{2}}\\y'''_{n-1}\\y'''_{n-\frac{3}{2}}\\y'''_{n-2}\\y'''_{n-\frac{5}{2}}\\y'''_{n-3}\\y'''_{n-\frac{7}{2}}\\y'''_n\end{bmatrix}+h^4\begin{bmatrix}0&0&0&0&0&0&0&\frac{24396497}{15328051200}\\0&0&0&0&0&0&0&\frac{1035731}{598752002}\\0&0&0&0&0&0&0&\frac{4104531}{63078400}\\0&0&0&0&0&0&0&\frac{76168}{467775}\\0&0&0&0&0&0&0&\frac{201421625}{613122048}\\0&0&0&0&0&0&0&\frac{142929}{246400}\\0&0&0&0&0&0&0&\frac{2048300303}{2189721600}\\0&0&0&0&0&0&0&\frac{660736}{467775}\end{bmatrix}\begin{bmatrix}f_{n-\frac{1}{2}}\\f_{n-1}\\f_{n-\frac{3}{2}}\\f_{n-2}\\f_{n-\frac{5}{2}}\\f_{n-3}\\f_{n-\frac{7}{2}}\\f_n\end{bmatrix} \tag{46}$$

$$+h^4\begin{bmatrix}\frac{1520909}{638668800}&\frac{-13220819}{3832012800}&\frac{8390797}{19600640}&\frac{-2050007}{510935040}&\frac{4854761}{1916006400}&\frac{-364589}{348364800}&\frac{162689}{63866880}&\frac{-42511}{15328051200}\\\frac{169969}{3742200}&\frac{-37379}{712800}&\frac{245837}{3742200}&\frac{357779}{5987520}&\frac{46873}{1247400}&\frac{-231691}{1496880}&\frac{14071}{3742200}&\frac{-8159}{19958400}\\\frac{156411}{716800}&\frac{-3119229}{15769600}&\frac{59337}{225280}&\frac{-1516887}{6307840}&\frac{1194183}{7884800}&\frac{-984537}{15769600}&\frac{17091}{1126400}&\frac{-20817}{12615680}\\\frac{96704}{155925}&\frac{-3119229}{15769600}&\frac{320704}{467775}&\frac{-50}{81}&\frac{181952}{467775}&\frac{-75032}{467775}&\frac{1216}{31185}&\frac{-1984}{467775}\\\frac{103514375}{76640256}&\frac{-41912}{93555}&\frac{111998125}{76640256}&\frac{-383340625}{366561024}&\frac{2901125}{3649536}&\frac{-50269375}{153280512}&\frac{79375}{995328}&\frac{-1773125}{204374016}\\\frac{38737}{15400}&\frac{-77193}{61600}&\frac{42057}{15400}&\frac{-53217}{24640}&\frac{21951}{15400}&\frac{-5139}{8800}&\frac{2187}{15400}&\frac{-3807}{246400}\\\frac{38267583}{91238400}&\frac{-89766187}{49766400}&\frac{1268609167}{273715200}&\frac{-244827569}{72990720}&\frac{650716619}{273715200}&\frac{-512596693}{547430400}&\frac{21001547}{91238400}&\frac{54841241}{2189721600}\\\frac{3043228}{467775}&\frac{-381952}{155925}&\frac{684032}{93555}&\frac{-302946223}{47900160}&\frac{5348}{14175}&\frac{-631808}{467775}&\frac{167936}{467775}&\frac{-1184}{31185}\end{bmatrix}\begin{bmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\\f_{n+\frac{5}{2}}\\f_{n+3}\\f_{n+\frac{7}{2}}\\f_{n+4}\end{bmatrix}$$

Writing out (46) explicitly as follow

$$y_{n+\frac{1}{2}} = y_n - \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + h^4\left(\begin{array}{l} \frac{24396497}{15328051200}f_n + \frac{1520909}{638668800}f_{n+\frac{1}{2}} - \frac{13220819}{3832012800}f_{n+1} \\ +\frac{8390797}{191600640}f_{n+\frac{3}{2}} - \frac{2050007}{510935040}f_{n+2} + \frac{4854761}{1916006400}f_{n+\frac{5}{2}} \\ -\frac{364589}{348364800}f_{n+3} + \frac{162689}{638668800}f_{n+\frac{7}{2}} - \frac{425111}{15328051200}f_{n+4} \end{array}\right) \tag{47}$$

$$y_{n+1} = y_n - hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + h^4\left(\begin{array}{l} \frac{1035731}{59875200}f_n + \frac{169969}{3742200}f_{n+\frac{1}{2}} - \frac{37379}{712800}f_{n+1} \\ +\frac{245837}{3742200}f_{n+\frac{3}{2}} - \frac{357779}{5987520}f_{n+2} + \frac{46873}{1247400}f_{n+\frac{5}{2}} \\ -\frac{231691}{1496880}f_{n+3} + \frac{14071}{3742200}f_{n+\frac{7}{2}} - \frac{8159}{19958400}f_{n+4} \end{array}\right) \tag{48}$$

$$y_{n+\frac{3}{2}} = y_n - \frac{3}{2}hy'_n + \frac{9}{8}h^2y''_n + \frac{9}{16}h^3y'''_n + h^4\left(\begin{array}{l} \frac{4104531}{63078400}f_n + \frac{156411}{716800}f_{n+\frac{1}{2}} - \frac{3119229}{15769600}f_{n+1} \\ +\frac{59337}{225280}f_{n+\frac{3}{2}} - \frac{1516887}{6307840}f_{n+2} + \frac{1194183}{7884800}f_{n+\frac{5}{2}} \\ -\frac{984537}{15769600}f_{n+3} + \frac{17091}{1126400}f_{n+\frac{7}{2}} - \frac{20817}{12615680}f_{n+4} \end{array}\right) \tag{49}$$

$$y_{n+2} = y_n - 2hy'_n + 2h^2y''_n + \frac{4}{3}h^3y'''_n + h^4\left(\begin{array}{l} \frac{76168}{467775}f_n + \frac{96704}{155925}f_{n+\frac{1}{2}} - \frac{41912}{93555}f_{n+1} \\ +\frac{320704}{467775}f_{n+\frac{3}{2}} - \frac{50}{81}f_{n+2} + \frac{181952}{467775}f_{n+\frac{5}{2}} \\ -\frac{75032}{467775}f_{n+3} + \frac{1216}{31185}f_{n+\frac{7}{2}} - \frac{1984}{467775}f_{n+4} \end{array}\right) \tag{50}$$

$$y_{n+\frac{5}{2}} = y_n - \frac{5}{2}hy'_n + \frac{25}{8}h^2y''_n + \frac{125}{48}h^3y'''_n + h^4\left(\begin{array}{l} \frac{201421625}{613122048}f_n + \frac{103514375}{76640256}f_{n+\frac{1}{2}} - \frac{40900625}{51093504}f_{n+1} \\ +\frac{111998125}{76640256}f_{n+\frac{3}{2}} - \frac{383340625}{306561024}f_{n+2} + \frac{2901125}{3649536}f_{n+\frac{5}{2}} \\ -\frac{50269375}{153280512}f_{n+3} + \frac{79375}{995328}f_{n+\frac{7}{2}} - \frac{1773125}{204374016}f_{n+4} \end{array}\right) \tag{51}$$

$$y_{n+3} = y_n - 3hy'_n + \frac{9}{2}h^2y''_n + \frac{9}{2}h^3y'''_n + h^4\left(\begin{array}{l} \frac{142929}{246400}f_n + \frac{38637}{15400}f_{n+\frac{1}{2}} - \frac{77193}{61600}f_{n+1} \\ +\frac{42057}{15400}f_{n+\frac{3}{2}} - \frac{53217}{24640}f_{n+2} + \frac{21951}{15400}f_{n+\frac{5}{2}} \\ -\frac{5139}{8800}f_{n+3} + \frac{2187}{15400}f_{n+\frac{7}{2}} - \frac{3807}{246400}f_{n+4} \end{array}\right) \tag{52}$$

$$y_{n+\frac{7}{2}} = y_n - \frac{7}{2}hy'_n + \frac{49}{8}h^2y''_n + \frac{343}{48}h^3y'''_n + h^4\begin{pmatrix} \frac{2048300303}{2189721600}f_n + \frac{38267583}{91238400}f_{n+\frac{1}{2}} - \frac{89766187}{49766400}f_{n+1} \\ +\frac{1268609167}{273715200}f_{n+\frac{3}{2}} - \frac{244827569}{72990720}f_{n+2} + \frac{650716619}{273715200}f_{n+\frac{5}{2}} \\ -\frac{512596693}{547430400}f_{n+3} + \frac{21001547}{91238400}f_{n+\frac{7}{2}} - \frac{54841241}{2189721600}f_{n+4} \end{pmatrix} \quad (53)$$

$$y_{n+4} = y_n - 4hy'_n + 8h^2y''_n + \frac{32}{3}h^3y'''_n + h^4\begin{pmatrix} \frac{660736}{467775}f_n + \frac{3043228}{467775}f_{n+\frac{1}{2}} - \frac{381952}{155925}f_{n+1} \\ +\frac{684032}{93555}f_{n+\frac{3}{2}} - \frac{302946223}{47900160}f_{n+2} + \frac{5348}{14175}f_{n+\frac{5}{2}} \\ -\frac{631808}{467775}f_{n+3} + \frac{167936}{467775}f_{n+\frac{7}{2}} - \frac{1184}{31185}f_{n+4} \end{pmatrix} \quad (54)$$

substituting the equation (47) - (54) into (20) - (27) gives

$$y'_{n+\frac{1}{2}} = -y'_n + \frac{1}{2}hy''_n + \frac{1}{8}h^2y'''_n + h^3\begin{pmatrix} \frac{517129}{45619200}f_n + \frac{1129981}{53222400}f_{n+\frac{1}{2}} - \frac{1871827}{63866880}f_{n+1} \\ +\frac{5887073}{159667200}f_{n+\frac{3}{2}} - \frac{716363}{21288960}f_{n+2} + \frac{3385541}{159667200}f_{n+\frac{5}{2}} \\ -\frac{2792861}{319334400}f_{n+3} + \frac{22637}{10644480}f_{n+\frac{7}{2}} - \frac{36943}{159667200}f_{n+4} \end{pmatrix} \quad (55)$$

$$y'_{n+1} = -y'_n + hy''_n + \frac{1}{2}h^2y'''_n + h^3\begin{pmatrix} \frac{286967}{4989600}f_n + \frac{32543}{178200}f_{n+\frac{1}{2}} - \frac{22063}{118800}f_{n+1} \\ +\frac{58657}{249480}f_{n+\frac{3}{2}} - \frac{21359}{99792}f_{n+2} + \frac{55969}{415800}f_{n+\frac{5}{2}} \\ -\frac{138317}{2494800}f_{n+3} + \frac{16799}{1247400}f_{n+\frac{7}{2}} - \frac{487}{332640}f_{n+4} \end{pmatrix} \quad (56)$$

$$y'_{n+\frac{3}{2}} = -y'_n + \frac{3}{2}hy''_n + \frac{9}{8}h^2y'''_n + h^3\begin{pmatrix} \frac{68769}{492800}f_n + \frac{1067877}{1971200}f_{n+\frac{1}{2}} - \frac{1563651}{3942400}f_{n+1} \\ +\frac{32841}{56320}f_{n+\frac{3}{2}} - \frac{419931}{788480}f_{n+2} + \frac{661959}{1971200}f_{n+\frac{5}{2}} \\ -\frac{546129}{3942400}f_{n+3} + \frac{66393}{1971200}f_{n+\frac{7}{2}} - \frac{2889}{788480}f_{n+4} \end{pmatrix} \quad (57)$$

$$y'_{n+2} = -y'_n + 2hy''_n + 2h^2y'''_n + h^3\begin{pmatrix} \frac{80293}{311850}f_n + \frac{57136}{51975}f_{n+\frac{1}{2}} - \frac{18812}{31185}f_{n+1} \\ +\frac{179408}{155925}f_{n+\frac{3}{2}} - \frac{296}{297}f_{n+2} + \frac{98096}{155925}f_{n+\frac{5}{2}} \\ -\frac{40468}{155925}f_{n+3} + \frac{656}{10395}f_{n+\frac{7}{2}} - \frac{2141}{311850}f_{n+4} \end{pmatrix} \quad (58)$$

$$y'_{n+\frac{5}{2}} = -y'_n + \frac{5}{2}hy''_n + \frac{25}{8}h^2y'''_n + h^3\begin{pmatrix} \frac{5253125}{12773367}f_n + \frac{11851375}{6386688}f_{n+\frac{1}{2}} - \frac{3432125}{4257792}f_{n+1} \\ +\frac{12768625}{6386688}f_{n+\frac{3}{2}} - \frac{19680625}{12773376}f_{n+2} + \frac{307625}{304128}f_{n+\frac{5}{2}} \\ -\frac{5327125}{12773376}f_{n+3} + \frac{647875}{6386688}f_{n+\frac{7}{2}} - \frac{5875}{532224}f_{n+4} \end{pmatrix} \quad (59)$$

$$y'_{n+3} = -y'_n + 3hy''_n + \frac{9}{2}h^2y'''_n + h^3\begin{pmatrix} \frac{37017}{61600}f_n + \frac{6183}{2200}f_{n+\frac{1}{2}} - \frac{6183}{6160}f_{n+1} \\ +\frac{48141}{15400}f_{n+\frac{3}{2}} - \frac{12933}{6160}f_{n+2} + \frac{23787}{15400}f_{n+\frac{5}{2}} \\ -\frac{42691}{4400}f_{n+3} + \frac{459}{3080}f_{n+\frac{7}{2}} - \frac{999}{61600}f_{n+4} \end{pmatrix} \quad (60)$$

$$y'_{n+\frac{7}{2}} = -y'_n + \frac{7}{2}hy''_n + \frac{49}{8}h^2y'''_n + h^3\begin{pmatrix} \frac{18850937}{22809600}f_n + \frac{30139753}{7603200}f_{n+\frac{1}{2}} - \frac{54639557}{45619200}f_{n+1} \\ +\frac{20689417}{4561920}f_{n+\frac{3}{2}} - \frac{1630279}{608256}f_{n+2} + \frac{52421033}{22809600}f_{n+\frac{5}{2}} \\ -\frac{35782103}{45619200}f_{n+3} + \frac{1570597}{7603200}f_{n+\frac{7}{2}} - \frac{40817}{1824768}f_{n+4} \end{pmatrix} \quad (61)$$

$$y'_{n+4} = -y'_n + 4hy''_n + 8h^2y'''_n + h^3\begin{pmatrix} \frac{24232}{22275}f_n + \frac{828928}{155925}f_{n+\frac{1}{2}} - \frac{72064}{51975}f_{n+1} \\ +\frac{194048}{31185}f_{n+\frac{3}{2}} - \frac{102112}{31185}f_{n+2} + \frac{169472}{51975}f_{n+\frac{5}{2}} \\ -\frac{134528}{155925}f_{n+3} + \frac{51712}{155925}f_{n+\frac{7}{2}} - \frac{296}{10395}f_{n+4} \end{pmatrix} \quad (62)$$

Substituting equation (47) - (54) into (29) – (36) gives

$$y''_{n+\frac{1}{2}} = y''_n + \tfrac{1}{2}hy'''_n + h^2\begin{pmatrix} \frac{324901}{5806080}f_n + \frac{8183}{57600}f_{n+\frac{1}{2}} - \frac{653203}{3628800}f_{n+1} + \\ \frac{50689}{226800}f_{n+\frac{3}{2}} - \frac{196277}{967680}f_{n+2} + \frac{92473}{725760}f_{n+\frac{5}{2}} \\ -\frac{95167}{1814400}f_{n+3} + \frac{7703}{604800}f_{n+\frac{7}{2}} - \frac{5741}{4147200}f_{n+4} \end{pmatrix} \quad (63)$$

$$y''_{n+1} = y''_n + hy'''_n + h^2\begin{pmatrix} \frac{58193}{453600}f_n + \frac{7346}{14175}f_{n+\frac{1}{2}} - \frac{81}{200}f_{n+1} \\ +\frac{7729}{14175}f_{n+\frac{3}{2}} - \frac{22703}{45360}f_{n+2} + \frac{1492}{4725}f_{n+\frac{5}{2}} \\ -\frac{14773}{113400}f_{n+3} + \frac{449}{14175}f_{n+\frac{7}{2}} - \frac{521}{151200}f_{n+4} \end{pmatrix} \quad (64)$$

$$y''_{n+\frac{3}{2}} = y''_n + \tfrac{3}{2}hy'''_n + h^2\left(\begin{array}{l}\frac{71661}{358400}f_n + \frac{1467}{1600}f_{n+\frac{1}{2}} - \frac{4707}{11200}f_{n+1} + \\ \frac{225}{256}f_{n+\frac{3}{2}} - \frac{28143}{35840}f_{n+2} + \frac{11079}{22400}f_{n+\frac{5}{2}} \\ -\frac{9141}{44800}f_{n+3} + \frac{2223}{44800}f_{n+\frac{7}{2}} - \frac{387}{71680}f_{n+4}\end{array}\right) \tag{65}$$

$$y''_{n+2} = y''_n + 2hy'''_n + h^2\left(\begin{array}{l}\frac{7703}{28350}f_n + \frac{6208}{4725}f_{n+\frac{1}{2}} - \frac{232}{567}f_{n+1} \\ +\frac{20032}{14175}f_{n+\frac{3}{2}} - \frac{47}{45}f_{n+2} + \frac{9536}{14175}f_{n+\frac{5}{2}} \\ -\frac{3944}{14175}f_{n+3} + \frac{64}{28350}f_{n+\frac{7}{2}} - \frac{209}{28350}f_{n+4}\end{array}\right) \tag{66}$$

$$y''_{n+\frac{5}{2}} = y''_n + \tfrac{5}{2}hy'''_n + h^2\left(\begin{array}{l}\frac{56975}{165888}f_n + \frac{248375}{145152}f_{n+\frac{1}{2}} - \frac{19375}{48384}f_{n+1} + \\ \frac{5143375}{72576}f_{n+\frac{3}{2}} - \frac{641875}{580608}f_{n+2} + \frac{225}{256}f_{n+\frac{5}{2}} \\ -\frac{12875}{36288}f_{n+3} + \frac{3125}{36288}f_{n+\frac{7}{2}} - \frac{3625}{387072}f_{n+4}\end{array}\right) \tag{67}$$

$$y''_{n+3} = y''_n + 3hy'''_n + h^2\left(\begin{array}{l}\frac{93}{224}f_n + \frac{369}{175}f_{n+\frac{1}{2}} - \frac{549}{1400}f_{n+1} + \frac{444}{175}f_{n+\frac{3}{2}} - \\ \frac{639}{560}f_{n+2} + \frac{9}{7}f_{n+\frac{5}{2}} - \frac{81}{200}f_{n+3} + \frac{18}{175}f_{n+\frac{7}{2}} - \frac{9}{800}f_{n+4}\end{array}\right) \tag{68}$$

$$y''_{n+\frac{7}{2}} = y''_n + \tfrac{7}{2}hy'''_n + h^2\left(\begin{array}{l}\frac{2019731}{4147200}f_n + \frac{216433}{86400}f_{n+\frac{1}{2}} - \frac{98441}{259200}f_{n+1} + \\ \frac{1601467}{518400}f_{n+\frac{3}{2}} - \frac{160867}{138240}f_{n+2} + \frac{55223}{32400}f_{n+\frac{5}{2}} \\ -\frac{127253}{518400}f_{n+3} + \frac{8183}{57600}f_{n+\frac{7}{2}} - \frac{57281}{4147200}f_{n+4}\end{array}\right) \tag{69}$$

$$y''_{n+4} = y''_n + 4hy'''_n + h^2\left(\begin{array}{l}\frac{7912}{14175}f_n + \frac{5888}{2025}f_{n+\frac{1}{2}} - \frac{1856}{4725}f_{n+1} + \frac{10496}{2835}f_{n+\frac{3}{2}} - \\ \frac{3632}{2835}f_{n+2} + \frac{10496}{4725}f_{n+\frac{5}{2}} - \frac{1856}{14175}f_{n+3} + \frac{5888}{14175}f_{n+\frac{7}{2}}\end{array}\right) \tag{70}$$

Substituting equation (47) - (54) into (38) – (45) gives

$$y'''_{n+\frac{1}{2}} = y'''_n + h\left(\begin{array}{l}\frac{1070017}{7257600}f_n + \frac{2233547}{3628800}f_{n+\frac{1}{2}} - \frac{2302297}{3628800}f_{n+1} + \frac{2797679}{3628800}f_{n+\frac{3}{2}} - \frac{31457}{45360}f_{n+2} + \\ \frac{1573169}{3628800}f_{n+\frac{5}{2}} - \frac{645607}{3628800}f_{n+3} + \frac{156437}{3628800}f_{n+\frac{7}{2}} - \frac{33953}{7257600}f_{n+4}\end{array}\right) \tag{71}$$

$$y'''_{n+1}=y'''_{n}+h\begin{pmatrix}\frac{32377}{226800}f_n+\frac{22823}{28350}f_{n+\frac{1}{2}}-\frac{21247}{113400}f_{n+1}+\frac{15011}{28350}f_{n+\frac{3}{2}}-\frac{2903}{5670}f_{n+2}+\\ \frac{9341}{28350}f_{n+\frac{5}{2}}-\frac{15577}{113400}f_{n+3}+\frac{953}{28350}f_{n+\frac{7}{2}}-\frac{119}{32400}f_{n+4}\end{pmatrix} \quad (72)$$

$$y'''_{n+\frac{3}{2}}=y'''_{n}+h\begin{pmatrix}\frac{12881}{89600}f_n+\frac{35451}{44800}f_{n+\frac{1}{2}}+\frac{719}{44800}f_{n+1}+\frac{39967}{44800}f_{n+\frac{3}{2}}-\frac{351}{560}f_{n+2}+\\ \frac{17217}{44800}f_{n+\frac{5}{2}}-\frac{7031}{44800}f_{n+3}+\frac{243}{6400}f_{n+\frac{7}{2}}-\frac{369}{89600}f_{n+4}\end{pmatrix} \quad (73)$$

$$y'''_{n+2}=y'''_{n}+h\begin{pmatrix}\frac{4063}{28350}f_n+\frac{11288}{14175}f_{n+\frac{1}{2}}+\frac{122}{14175}f_{n+1}+\frac{16376}{14175}f_{n+\frac{3}{2}}-\frac{908}{2835}f_{n+2}+\\ \frac{4616}{14175}f_{n+\frac{5}{2}}-\frac{1978}{14175}f_{n+3}+\frac{488}{14175}f_{n+\frac{7}{2}}-\frac{107}{28350}f_{n+4}\end{pmatrix} \quad (74)$$

$$y'''_{n+\frac{5}{2}}=y'''_{n}+h\begin{pmatrix}\frac{41705}{290304}f_n+\frac{115075}{145152}f_{n+\frac{1}{2}}+\frac{3775}{145152}f_{n+1}+\frac{159175}{145152}f_{n+\frac{3}{2}}-\frac{125}{9072}f_{n+2}+\\ \frac{85465}{145152}f_{n+\frac{5}{2}}-\frac{24575}{145152}f_{n+3}+\frac{5725}{145152}f_{n+\frac{7}{2}}-\frac{175}{41472}f_{n+4}\end{pmatrix} \quad (75)$$

$$y'''_{n+3}=y'''_{n}+h\begin{pmatrix}\frac{401}{2800}f_n+\frac{279}{350}f_{n+\frac{1}{2}}+\frac{9}{1400}f_{n+1}+\frac{403}{350}f_{n+\frac{3}{2}}-\frac{9}{70}f_{n+2}+\\ \frac{333}{350}f_{n+\frac{5}{2}}+\frac{79}{1400}f_{n+3}+\frac{9}{350}f_{n+\frac{7}{2}}-\frac{9}{2800}f_{n+4}\end{pmatrix} \quad (76)$$

$$y'''_{n+\frac{7}{2}}=y'''_{n}+h\begin{pmatrix}\frac{149527}{1036800}f_n+\frac{408317}{518400}f_{n+\frac{1}{2}}+\frac{24353}{518400}f_{n+1}+\frac{542969}{518400}f_{n+\frac{3}{2}}+\frac{343}{6480}f_{n+2}+\\ \frac{368039}{518400}f_{n+\frac{5}{2}}+\frac{261023}{518400}f_{n+3}+\frac{111587}{518400}f_{n+\frac{7}{2}}-\frac{8183}{1036800}f_{n+4}\end{pmatrix} \quad (77)$$

$$y'''_{n+4}=y'''_{n}+h\begin{pmatrix}\frac{1978}{14175}f_n+\frac{11776}{14175}f_{n+\frac{1}{2}}-\frac{1856}{14175}f_{n+1}+\frac{20992}{14175}f_{n+\frac{3}{2}}-\frac{21816}{2835}f_{n+2}+\\ \frac{20992}{14175}f_{n+\frac{5}{2}}-\frac{1856}{14175}f_{n+3}+\frac{11776}{14175}f_{n+\frac{7}{2}}+\frac{1978}{14175}f_{n+4}\end{pmatrix} \quad (78)$$

3. Basic Properties of the Methods

Here, we present the basic properties of the new numerical method which include order and error constant, consistency, convergence and zero stability.

3.1. Consistency of the Numerical Method

A numerical method is said to be consistence if the following are satisfies.

i. The order of the scheme must be greater than or equal to 1 i.e. p≥1.

ii. $\sum_{j=0}^{k} \alpha_j = 0, \alpha_j$'s are the coefficients of the first characteristic polynomial

iii. $\rho(r) = \rho'(r) = 0$ where $r = 1$

iv. $\rho^{\mathbf{iv}}(r) = 4!\sigma(r)$ for $r = 1$, σ (r) is the second characteristic polynomial.

According to [7] the first condition is a sufficient condition for the associated block method to be consistent. To obtain the consistency, we used the above conditions;

3.2. Consistency of the Proposed Method

Consider the new developed method (11)

$$y_{n+4} + 140y_{n+1} - 120y_{n+\frac{1}{2}} - 56y_{n+\frac{3}{2}} + 35y_n = \frac{h^4}{829440}\begin{bmatrix} -565f_n + 298040f_{n+\frac{1}{2}} + 1409780f_{n+1} + \\ 1022792f_{n+\frac{3}{2}} + 615410f_{n+2} + 192200f_{n+\frac{5}{2}} \\ +86420f_{n+3} + 4280f_{n+\frac{7}{2}} + 443f_{n+4} \end{bmatrix} \quad (79)$$

i. The order, p=10 hence it satisfies p⩾1

ii. $\sum_{j=0}^{4} \alpha_j = \alpha_0 + \alpha_{\frac{1}{2}} + \alpha_1 + \alpha_{\frac{3}{2}} + \alpha_4 = 35 - 120 + 140 - 56 + 1 = 0$

iii. $\rho(r) = r^4 + 140r^1 - 120r^{\frac{1}{2}} - 56r^{\frac{3}{2}} + 35r^0$

$$\rho(r) = r^4 + 140r^1 - 120r^{\frac{1}{2}} - 56r^{\frac{3}{2}} + 35$$

$$\rho(1) = (1)^4 + 140(1)^1 - 120(1)^{\frac{1}{2}} - 56(1)^{\frac{3}{2}} + 35 \overset{?}{=} 0$$

$$\rho'(r) = 4r^3 + 140 - 60r^{\frac{-1}{2}} - 84r^{-\frac{1}{2}}$$

$$\rho'(1) = (1)^3 + 140 - 60(1)^{\frac{-1}{2}} - 84(1)^{-\frac{1}{2}} \overset{?}{=} 0$$

Since p (1)= $\rho'(r) = 0$, this shows that condition is satisfied

iv. By the main method, the first and second characteristics polynomial of the method are

$$p(r) = r^4 + 140r^1 - 120r^{\frac{1}{2}} - 56r^{\frac{3}{2}} + 35r^0 = r^4 + 140r^1 - 120r^{\frac{1}{2}} - 56r^{\frac{3}{2}} + 35$$

$$\sigma(r) = \frac{-565r^0}{829440} + \frac{298040r^{\frac{1}{2}}}{829440} + \frac{1409780r^1}{829440} + \frac{1022792r^{\frac{3}{2}}}{829440} + \frac{615410r^2}{829440} + \frac{192200r^{\frac{5}{2}}}{829440} + \frac{86420r^3}{829440} + \frac{4280r^{\frac{7}{2}}}{829440} + \frac{443r^4}{829440}$$

$$= \frac{-565}{829440} + \frac{298040r^{\frac{1}{2}}}{829440} + \frac{1409780r^1}{829440} + \frac{1022792r^{\frac{3}{2}}}{829440} + \frac{615410r^2}{829440} + \frac{192200r^{\frac{5}{2}}}{829440} + \frac{86420r^3}{829440} + \frac{4280r^{\frac{7}{2}}}{829440} + \frac{443r^4}{829440}$$

$$\rho'(r) = 4r^3 + 140 - 60r^{\frac{-1}{2}} - 84r^{-\frac{1}{2}}$$

$$\rho''(r) = 12r^2 + 30r^{\frac{-3}{2}} - 42r^{\frac{-1}{2}}$$

$$p(r)''' = 24r - 45r^{\frac{-5}{2}} + 21r^{\frac{-3}{2}}$$

$$p(r)^{\mathbf{iv}} = 24 + \frac{225}{2}(r)^{-\frac{7}{2}} - \frac{63}{2}(r)^{\frac{-5}{2}}$$

Putting r=1

$$p(1)^{\mathbf{iv}} = 24 + \frac{225}{2}(1)^{-\frac{7}{2}} - \frac{63}{2}(1)^{\frac{-5}{2}} = 105$$

$$\sigma(r) = \frac{-565}{829440} + \frac{298040r^{\frac{1}{2}}}{829440} + \frac{1409780r^{1}}{829440} + \frac{1022792r^{\frac{3}{2}}}{829440} + \frac{615410r^{2}}{829440} + \frac{192200r^{\frac{5}{2}}}{829440} + \frac{86420r^{3}}{829440} + \frac{4280r^{\frac{7}{2}}}{829440} + \frac{443r^{4}}{829440}$$

$$\sigma(1) = \frac{1}{829440}[-565 + 298040 + 1409780 + 1022792 + 615410 + 192200 + 86420 + 4280 + 443] = \frac{3628800}{8299440}$$

$$4!\sigma(1) = 4!*\frac{4375}{1000} = 105$$

$$p(1)^{\mathbf{iv}} = 4!\sigma(1) = 105$$

For principal root r=1, the conditions (1)-(iv) above are satisfied. Hence the Method is Consistence.

3.3. Zero Stability of Block Method

Given the general form of block method as

$$A^{(0)}Y_m = A^{(i)}Y_{m-1} + h^{\mu}[B^{(i)}F_m + B^{(0)}F_{m-1}]$$

A block method is said to be zero stable, if the roots

$$\det\left[\lambda A^{(0)} - A^{(i)}\right] = 0$$

of the first characteristic polynomial satisfy $|\lambda| \leq 1$ and for the roots with $|\lambda| \leq 1$, the multiplicity must not exceed the order of the differential equation.

3.4. Zero Stability of the Proposed Block Method

Given the general form of block method as

$$A = \left[z \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \right] = 0$$

$$z = 0,0,0,0,0,0,0,1$$

Hence the block is zero stable.

3.5. Convergence

Theorem 1: Convergence [10-11]

The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable. From the theorem above, the blocks are convergent.

4. Numerical Experiment

The mode of implementation of our method is by combining the block integrators (47 - 78) to simultaneously solve the system of equations of initial value problems of fourth order Ordinary differential equations without requiring starting values and predictors.

4.1. Numerical Example

Consider the linear system below, solved by erudite scholar Kasim*et al* (2015)

$$y^{iv} = e^{3x}u, \quad y(0)=1, y'(0)=-1, y''(0)=1, y'''(0)=-1$$
$$z^{iv} = 16e^{-x}y, \quad z(0)=1, z'(0)=-2, z''(0)=4, z'''(0)=-8$$
$$w^{iv} = 81e^{-x}z, \quad w(0)=1, w'(0)=-3, w''(0)=9, w'''(0)=-27$$
$$u^{iv} = 256e^{-x}w, \quad u(0)=1, u'(0)=-4, u''(0)=16, u'''(0)=-64$$

The exact solution is given by

$$y = e^{-x} \qquad z = e^{-2x}$$
$$w = e^{-3x} \qquad u = e^{-4x}$$

The problem is integrated in the interval $[0,3]$

4.2. Numerical Results

The numerical results of the proposed method are presented in Table 1 – Table 4 below.

Table 1. *The y-exact Result, y-computed results and absolute error in the new method.*

x	y-Exact Result	y-Computed Result	y-Error	y-Time
0.1	0.9048374180359595	0.904837418035959520	0.000000e+00	0.0014
0.2	0.8187307530779818	0.818730753077981930	1.110223e-16	0.0020
0.3	0.7408182206817178	0.740818220681717880	1.110223e-16	0.0022
0.4	0.6703200460356393	0.670320046035639330	0.000000e+00	0.0023
0.5	0.6065306597126334	0.606530659712633540	1.110223e-16	0.0024
0.6	0.5488116360940264	0.548811636094026390	0.000000e+00	0.0026

x	y-Exact Result	y-Computed Result	y-Error	y-Time
0.7	0.4965853037914095	0.496585303791409580	1.110223e-16	0.0027
0.8	0.4493289641172216	0.449328964117221900	3.330669e-16	0.0028
0.9	0.4065696597405991	0.406569659740599500	3.885781e-16	0.0029
1.0	0.3678794411714423	0.367879441171442890	5.551115e-16	0.0031

Table 2. *The z-exact solutions, z-computed results and absolute error in the developed method.*

x	z-Exact Result	z-Computed Result	z-Error	z-Time
0.1	0.8187307530779818	0.818730753077985710	3.885781e-15	0.0409
0.2	0.6703200460356393	0.670320046035695950	5.662137e-14	0.0445
0.3	0.5488116360940264	0.548811636094255100	2.287059e-13	0.0471
0.4	0.4493289641172216	0.449328964117811200	5.896394e-13	0.0498
0.5	0.3678794411714423	0.367879441172651030	1.208700e-12	0.0525
0.6	0.3011942119122020	0.301194211914357360	2.155331e-12	0.0556
0.7	0.2465969639416064	0.246596963945105550	3.499118e-12	0.0579
0.8	0.2018965179946554	0.201896517999964550	5.309170e-12	0.0613
0.9	0.1652988882215865	0.165298888229241880	7.655349e-12	0.0637
1.0	0.1353352832366127	0.135335283247220110	1.060740e-11	0.0660

Table 3. *The w-exact solutions, w-computed results and absolute error in the developed method.*

x	w-Exact Result	w-Computed Result	w-Error	w-Time
0.1	0.7408182206817178	0.740818220682830760	1.112999e-12	0.0407
0.2	0.5488116360940264	0.548811636109987180	1.596079e-11	0.0444
0.3	0.4065696597405991	0.406569659805117440	6.451839e-11	0.0471
0.4	0.3011942119122020	0.301194212078471410	1.662694e-10	0.0493
0.5	0.2231301601484298	0.223130160489256490	3.408267e-10	0.0529
0.6	0.1652988882215865	0.165298888829352730	6.077662e-10	0.0556
0.7	0.1224564282529819	0.122456429239647950	9.866660e-10	0.0579
0.8	0.0907179532894125	0.090717954786547317	1.497135e-09	0.0606
0.9	0.0672055127397498	0.067205514898420304	2.158671e-09	0.0630
1.0	0.0497870683678639	0.049787071359048696	2.991185e-09	0.0661

Table 4. *The u-exact solutions, u-computed results and absolute error in the developed method.*

x	u-Exact Result	u-Computed Result	u-Error	u-Time
0.1	0.6703200460356393	0.670320046089736500	5.409717e-11	0.0423
0.2	0.4493289641172216	0.449328964892583670	7.753621e-10	0.0459
0.3	0.3011942119122020	0.301194215046340250	3.134138e-09	0.0494
0.4	0.2018965179946554	0.201896526071894570	8.077239e-09	0.0518
0.5	0.1353352832366127	0.135335299794267390	1.655765e-08	0.0547
0.6	0.0907179532894125	0.090717982816007847	2.952660e-08	0.0575
0.7	0.0608100626252180	0.060810110560611363	4.793539e-08	0.0597
0.8	0.0407622039783662	0.040762276715137880	7.273677e-08	0.0646
0.9	0.0273237224472926	0.027323827325754557	1.048785e-07	0.0670
1.0	0.0183156388887342	0.018315784215550579	1.453268e-07	0.0697

5. Discussion of the New Method and Results

This work presents the derivation of a new numerical method for solving fourth order ordinary differential equation. The method makes use of power series as the approximate solution to (1). The method is fully implicit in nature with four-hybrid points which were carefully selected with equal interval within four-step. The method has order of accuracy P=10, using Taylor series as established in [7, 13]. The method is consistence, convergent and zero stable. The method was applied to solve linear system of equations. The Results were displayed in Table 1 – Table 4.

6. Conclusion and Recommendation of the New Method

In this study, we discussed in details, the derivation, and analysis of a new numerical method for solving fourth order ordinary differential equations by hybrid block techniques. Power series was used as basic function. The method has four grid points and 4-off grid points as it has been chosen carefully within the interval of integration of four-step. The main purpose is to introduce a new numerical method as a recommendation of numerical approximate values which may be agreed or tends to some existing method of solution of various initial value problems of fourth order ordinary differential equations. We have been able to establish the new proposed method by the Results in Table 1 – Table 4. All the mathematical formulations were carried out by using Maple Mathematical Software. In future, we will attempt to check the authenticity of this method on boundary value problem with numerous complex examples.

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