Some New Properties of $W_d$-fuzzy Implication Algebras

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Abstract: Implication is an important logical connective in practically every propositional logic. In 1987, the so-called Fuzzy implication algebras were introduced by Wu Wangming, then various interesting properties of FI-algebras and some subalgebra of Fuzzy implication algebra, such as regular FI-algebras, commutative FI-algebras, $W_d$-FI-algebras, and other kinds of FI-algebras were reported. The main aim of this article is to study $W_d$-fuzzy implication algebras which are subalgebra of fuzzy implication algebras. We showed that $W_d$ -fuzzy implication algebras are regular fuzzy implication algebras, but the inverse is not true. The relations between $W_d$-fuzzy implication algebras and other fuzzy algebras are discussed. Properties and axiomatic systems for $W_d$ -fuzzy implication algebras are investigated. Furthermore, a few new results on $W_d$ -fuzzy implication algebras has been added.

Keywords: Fuzzy Implication Algebras, $W_d$ -Fuzzy Implication, Regular Fuzzy Implication Algebras, Heyting Type Fuzzy Implication Algebras

1. Introduction

In the past years, fuzzy algebras and their axiomatization have become important topics in theoretical research and in the applications of fuzzy logic. The implication connective plays a crucial role in fuzzy logic and reasoning [2]. Recently, some authors studied fuzzy implications from different perspectives[16]. Naturally, it is meaningful investigating the common properties of some important fuzzy implications used in fuzzy logic. Consequently, Professor Wu [1] introduced a class of fuzzy implication algebras, FI-algebras for short, in 1990.

In the past two decades, some authors focused on FI-algebras. Various interesting properties of FI-algebras [3-5], regular FI-algebras [1,6,7], commutative FI-algebras [8], $W_d$-FI-algebras [9], and other kinds of FI-algebras [10] were reported, and some concepts of filter, ideal and fuzzy filter of FI-algebras were proposed [1,11,12]. Relationships between FI-algebra and BCK-algebra [13,14], MV-algebras [15], Rough set algebras [16,17], and were partly investigated, and FI-algebras were axiomatized [18]. In the recent work, the relationship between these FI- algebras and several famous fuzzy algebras were systematically discussed[3,19-24].

The organization of the paper is as follows: preliminary notions and results are introduced in section 2; section 3 relationships between $W_d$ -FI algebras and several classes of important fuzzy algebras are discussed; Main properties of $W_d$ -FI algebras is included in section 4. Lastly, the paper introduces several conclusions and pointers for further research.

2. Preliminaries

In this section,we summarize some definitions and results about $W_d$ -FI algebras,which will be used in the following sections of the paper.

First, we recall some definitions and properties about $W_d$ -FI algebras.

Definition 2.1. (see [1]) Let $X$ be a set with a binary operation $\rightarrow$, and $0 \in X$. An fuzzy implication algebra, shortlty,FI-algebra is an algebra $(X, \rightarrow, 0)$ of type(2,0)satisfying

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$;
(I2) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$;
(I3) $x \rightarrow x = 1$;
(I4) if $x \rightarrow y = y \rightarrow x = 1$, then $x = y$;
(I5) $0 \rightarrow z = 1$, for all $x, y, z \in X$, where $1 = 0 \rightarrow 0$. 

Definition 2.2. (see [1]). A Hilbert fuzzy implication algebra, or shortly, HFI-algebra, is an algebra \((X; u, 0, 0)\) of type \((2, 0)\) which satisfies the following conditions for every \(x, y, z \in X\):

\[(H1) \ x \to (y \to x) = 1;\]
\[(H2) \ (x \to (y \to z)) = ((x \to y) \to (x \to z));\]
\[(H3) \ x \to (y \to x) = 1;\]
\[(H4) \ x \to y = x \to 1, \text{ then } x = y;\]
\[(H5) \ 0 \to x = 1, \text{ where } 1 = 0 \to 0.\]

On an FI-algebra \(X\), one can define a binary relation \(\leq\) and operators \(C, T, S\) as follows:

\[x \leq y \iff x \to y = 1, x, y \in X;\]  
\[C(x) = x \to 0, x \in X;\]  
\[T(x, y) = C(x \to C(y));\]  
\[S(x, y) = C(x) \to y, x, y \in X.\]  

Usually, we also say that \(X\) is an FI-algebra for convenience.

Obviously, the relation \(\leq\) is a partial ordering on \(X\), i.e., the relation is reflexive, antisymmetric and transitive (see [1]).

In fuzzy logic, the property (1) is called the ordering property.

The operator \(C\) defined in the above definition is a negation on \(X\), i.e., the operator is order-inverting and satisfies \(C(0) = 1\) and \(C(1) = 0\). \(\leq\) and \(C\) are called the partial ordering and the negation induced by the FI-algebra \(X\), respectively.

If an FI-algebra \(X\) form a lattice with respect to the partial order \(\leq\), then we called FI-lattice.

Definition 2.3. (see [1,3]). Let \(X\) be an FI-algebra.

(i) \(X\) is regular FI-algebra, or an RFI-algebra, if \(CC(x) = x\), for all \(x \in X\).

(ii) \(X\) is commutative, or a CFI-algebra, if the binary operation \(\odot\) defined by \((3)\) is commutative, or the following condition \((6)\) holds for all \(x, y, z \in X\):

\[(I6) \ (x \odot y) = y \odot (y \to x) \to x.\]

Then \(W_d\) - Fuzzy implication algebra is an algebra of type \((2, 0)\). The notion was first formulated in 1996 by Deng and some properties were obtained (see [9]). This notion was originated from the motivation based on fuzzy implication algebra introduced by Wu (see [1]).

Definition 2.4. (see [9]). A \((2, 0)\)-type algebra \((X, \to, 0)\) is a \(W_d\)-Fuzzy implication algebra, or shortly, \(W_d\)-FI-algebra, if the following conditions hold for all \(x, y, z \in X\):

\[(W1) \ x \to (y \to z) = y \to (x \to z);\]
\[(W2) \ (x \to y) \to z = (z \to y) \to x;\]
\[(W3) \ x \to x = 1;\]
\[(W4) \ x \to y = x \to 1, \text{ then } x = y;\]
\[(W5) \ 0 \to x = 1, \text{ where } 1 = 0 \to 0.\]

Example 1 Consider \(X = [0, 1]\), for every \(x, y \in X\), defined \(x \to y = 1\), then \((X, \to, 0)\) is a \(W_d\)-FI algebras. Example 2 Let \(X = \{0, a, 1\}\) be a finite set of distinct elements. We define

\[\rightarrow 0 \quad 0 \quad a \quad 1\]
\[0 \quad 1 \quad 1\]
\[a \quad 0 \quad 1\]
\[1 \quad 0 \quad a \quad 1\]

Then \((X, \to, 0)\) is a FI-algebra, but not \(W_d\)-FI algebras. In fact, \((0 \to a) \to 0 = a \to 0 = 0\), but \((0 \to a) \to 1 = 1 \to 1 = 1\), so \((W2)\) does not hold.

3. Relationships Between \(W_d\)-FI Algebras and Two Classes of Important Fuzzy Algebras

Lemma 3.1. Let \((X, \to, 0)\) is a \(W_d\)-FI algebra, then for any \(x, y, z \in X\), the following holds:

\[(W6) \ x \to 1 = 1, \to x = x, \text{ for all } x \in X;\]
\[(W7) \text{ if } x \to y = 1, y \to z = 1, \text{ then } x \to z = 1, \text{ for all } x, y, z \in X;\]
\[(W8) \ (x \to y) \odot (y \to z) \odot (x \to z) = (x \to y) \odot (y \to z) \odot (y \to z) \odot (y \to z);\]
\[(W9) \ (x \to y) \odot (x \to z) = y \to z, (x \to y) \odot (y \to z) \odot (y \to z);\]
\[(W10) \ (x \to (y \to z)) \odot (x \to y) \odot (x \to z) = x.\]

Proof. (W6) Indeed \(x \to 1 = x \to (0 \to x) = 0 \to (x \to x) = 0 \to 1 = 1\). Thus, we have verified that \(x \to 1 = 1\).

Besides, \((1 \to x) \to x = (x \to x) \to 1 = 1 \to 1 = 1, x \to (1 \to x) = 1 \to (x \to x) = 1 \to 1 = 1\). that is \(1 \to x = x\).

(W7) If \(x \to y = 1, y \to z = 1\) holds, then \(x \to z = 1 \to (x \to z) = (x \to y) \odot (x \to z) = ((x \to z) \odot y) \odot x = ((y \to z) \odot x) \odot x = ((y \to z) \odot x);\)

Thus, \(x \to z = 1\).

(W8) Indeed \((x \to y) \odot (y \to z) \odot (x \to z) = (y \to z) \odot ((x \to y) \odot (x \to z)) = ((x \to y) \odot (y \to z) \odot (y \to z) \odot (y \to z));\)

The proof of \((W9), (W10)\) is similar to previous ones.

Lemma 3.2. Any \(W_d\)-FI algebra must be an FI-algebra, but the inverse is not true.

Proof. From the definition 2.3 and (3) of Lemma 3.1, it is easy to see that any \(W_d\)-FI algebra must be an FI-algebra. By example 2, \(W_d\)-FI algebra be an proper subalgebra of FI-algebra, but FI-algebra must be not \(W_d\)-FI algebra.

Theorem 3.1. Any \(W_d\)-FI algebra must be an RFI-algebra, but the inverse is not true (see [9]).

We have see that \(W_d\)-FI algebra classes are subclasses of RFI-algebras.

Theorem 3.2. \(W_d\)-FI algebra must be not CFI-algebra.

Proof. It is easy to proof that if \(x \neq y\), then the condition \((6)\) does not hold, i.e., suppose

\[(x \to y) \odot y = (y \to x) \to x,\]
then clearly
\[(y \to y) \to x = (x \to x) \to y.\]

Thus, we have \(x = y\), which contradicts to assertion.

**Theorem 3.3.** (see[9]) Relations between \(W_d\)-FI algebra and HFI - algebra as following:
1. If \((X, \to, 0)\) is a \(W_d\) -FI algebra such that, for all \(x, y, z \in X\),
\[x \to (y \to z) = (x \to y) \to (x \to z)\]
holds, then \((X, \to, 0)\) is a HFI-algebra.

2. If \((X, \to, 0)\) is a HFI-algebra such that, for all \(x, y, z \in X\),
\[(x \to y) \to z = (z \to y) \to x\]
holds, then \((X, \to, 0)\) is a \(W_d\) -FI algebra.

### 4. Main Properties of \(W_d\)-FI Algebras

On a \(W_d\) -FI algebra \(X\), one define a binary relation \(\leq\) as follows, \(x \leq y\) if and only if \(x \to y = 1, x, y \in X\).

Obviously, the relation “\(\leq\)" is a partial ordering on \(X\).

**Theorem 4.1.** Let \(X\) be a \(W_d\) -FI algebra, then exist a partial ordering in \(X\).

**Proof.** Indeed, \(\forall x, y \in X, x \to y = (1 \to x) \to y = (y \to x) \to 1 = 1\). Thus, we have verified that \(x \leq y\). Therefore, for any \(W_d\)-FI algebra must exist a partial ordering in \(X\).

**Theorem 4.2.** Let \(X\) be a \(W_d\) -FI algebra, and \(x, y, z \in X\). Then
\[
\begin{align*}
&W(1) \text{ If } x \leq y, \text{ then } z \to x \leq z \to y, y \to z \leq x \to z; \\
&W(12) x \leq CC(x); \\
&W(13) CCC(x) = C(x); \\
&W(14) C(x) \to y = C(y) \to x; \\
&W(15) \text{ (Commutativity) } T(x, y) = T(y, x), S(x, y) = S(y, x); \\
&W(16) \text{ (Associativity) } T(T(x, y), z) = T(x, T(y, z)), S(S(x, y), z) = S(x, S(y, z)); \\
&W(17) \text{ (Monotonicity) if } x \leq y, \text{ then } T(x, z) \leq T(y, z), S(x, z) \leq S(y, z); \\
&W(18) \text{ (Identity) } T(x, 1) = x, S(x, 0) = x; \\
&W(19) \text{ (Duality) } S(x, y) = C(T(C(x, C(y))), T(x, y)) = C(S(C(x), C(y)));
\end{align*}
\]
\[
\begin{align*}
&W(20) S(x, C(x)) = 1, S(x, C(x)) = 0; \\
&W(21) x \to (y \to z) = T(x, y) \to z; \\
&W(22) T((z \to x), (z \to y)) = z \to T(x, y); \\
&W(23) x \to x = x.
\end{align*}
\]

**Proof.** (W16) \(T(T(x, y), z) = T(C(x \to C(y)), z) = C(C(x \to C(y)) \to C(z)) = C((x \to C(y)) \to 0) \to C(z) = C(((x \to C(y)) \to 0) \to (z \to 0)) = C(z \to (C(y) \to (z \to 0))) = C((x \to z) \to (y \to z)) = T(x, T(y, z)).\)

Similarly, we have \(S(S(x, y), z) = S(x, S(y, z)).\)

(W17) Due to \(x \leq y \iff x \to y = 1\), then for all \(x, y, z \in X\), it is \(C(x \to C(z)) \to C(y \to C(z)) = ((x \to C(z)) \to 0) \to (y \to C(z)) = 0 \to (y \to C(z)) \to 0 = (y \to C(z)) \to 0 = (x \to C(z)) = x \to y = 1\). Hence, \(T(x, z) \leq T(y, z)\).

**Theorem 4.3.** Let \(X\) be a \(W_d\)-FI algebra, and \(x, y, z \in X\). Then
\[1 \to x = x, (x \to y) \to z = (z \to y) \to x \text{ imply } x \to (y \to z) = y \to (x \to z).
\]

**Proof.** (W21) \(x \to (y \to z) = (1 \to x) \to (y \to z) = ((y \to z) \to x) \to 1 = ((x \to z) \to y) \to 1 = (1 \to y) \to (x \to z) = y \to (x \to z).
\)

Hence, \(x \to (y \to z) = y \to (x \to z).
\]

**Theorem 4.4.** A (2,0)-type algebra \((X, \to, 0)\) is a \(W_d\)-fuzzy implication algebra if and only if it satisfies that
\[
\begin{align*}
&W(1') (x \to y) = (z \to y) \to x; \\
&W(2') 1 \to x = x; \\
&W(3') x \to x = 1; \\
&W(4') (x \to y) \to y = x = 1, \text{ then } x = y; \\
&W(5') 0 \to x = 1, \text{ where } 1 = 0 \to 0.
\end{align*}
\]

**Proof.** Immediate from theorem 4.2 and definition 2.3.

Condition (W3) and (W6) states that \(I\) is a logical unit and the greatest element of \(W_d\)-FI algebra. Note that a logical unit is always unique. We say that an \(W_d\)-FI algebra \(X\) has a negation if \(X\) admits a smallest element \(0\) such that the map: \(C : x \to C(x)\) is bijective, where \(C(x) = x \to 0\).

For an \(W_d\)-FI algebra with negation, we define two binary operation on \(X\) as follows: for any \(x, y \in X\),
\[
\begin{align*}
&x \perp y = C(x) \to y, \\
&x \top y = C(x \to C(y)).
\end{align*}
\]

**Theorem 4.5.** Let \(X\) be a \(W_d\)-FI algebra. Then for any \(x, y, z \in X\) we have:
1. \((x \perp y) \perp z = x \perp (y \top z); \)
2. \((x \perp y) \top z = x \perp (y \perp z); \)
3. \(x \perp 1 = 1, x \top 1 = x, x \top 0 = x; \)
4. \(x \top C(x) = 0, x \perp C(x) = 1; \)
5. \(x \perp y = C(C(x) \top C(y)), x \top y = C(C(x) \perp C(y)); \)
6. \(C(x) \to C(y) = y \to x, C(x) \to y = C(y) \to x; \)
under what conditions?

In future we will study the following topics:


