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Conference Paper

An Introduction to Differentiable Manifolds

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Abstract: A manifold is a Hausdorff topological space with some neighborhood of a point that looks like an open set in a Euclidean space. The concept of Euclidean space to a topological space is extended via suitable choice of coordinates. Manifolds are important objects in mathematics, physics and control theory, because they allow more complicated structures to be expressed and understood in terms of the well–understood properties of simpler Euclidean spaces. A differentiable manifold is defined either as a set of points with neighborhoods homeomorphic with Euclidean space, R^n with coordinates in overlapping neighborhoods being related by a differentiable transformation or as a subset of R, defined near each point by expressing some of the coordinates in terms of the others by differentiable functions. This paper aims at making a step by step introduction to differential manifolds.

Keywords: Submanifold, Differentiable Manifold, Morphism, Topological Space

1. Introduction

The concept of manifolds is central to many parts of geometry and modern mathematical physics because it allows more complicated structures to be expressed and understood in terms of the relatively well-understood properties of simpler spaces. Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold, [1], [2], [7]. For most applications, a special kind of topological manifold, a differentiable manifold, is used. If the local charts on a manifold are compatible in a certain sense, one can define directions, tangent spaces, and differentiable functions on that manifold. In particular it is possible to use calculus on a differentiable manifold. Each point of a differentiable manifold has a tangent space. This is a Euclidean space consisting of the tangent vectors of the curves through the point. Two important classes of differentiable manifolds are smooth and analytic manifolds. For smooth manifolds the transition maps are smooth, that is infinitely differentiable. Analytic manifolds are smooth manifolds with the additional condition that the transition maps are analytic. In other words, a differentiable (or, smooth) manifold is a topological manifold with a globally defined differentiable (or, smooth) structure, [1], [3], [4]. A topological manifold can be given a differentiable structure locally by using the homeomorphisms in the atlas of the topological space. The global differentiable structure is induced when it can be shown that the natural compositions of the homeomorphisms between the corresponding open Euclidean spaces are differentiable on overlaps of charts in the atlas. Therefore, the coordinates defined by the homeomorphisms are differentiable with respect to each other when treated as real valued functions with respect to the variables defined by other coordinate systems whenever charts overlap, [5]. This idea is often presented formally using transition maps. This allows one to extend the meaning of differentiability to spaces without global coordinate systems. Specifically, a differentiable structure allows one to define a global differentiable tangent space, and consequently, differentiable functions, and differentiable tensor–fields, [1], [2], [3], [11].

Differentiable manifolds are very important in physics. Special kinds of differentiable manifolds form the arena for physical theories such as classical mechanics, general relativity and Yang–Mills gauge theory. It is possible to develop calculus on differentiable manifolds, leading to such mathematical machinery as the exterior calculus. Historically, the development of differentiable manifolds is usually credited to C. F. Gauss and his student B. Riemann. The work of physicists J. C. Maxwell and A. Einstein lead to the development of the theory transformations between coordinate systems which preserved the essential geometric properties. Eventually these ideas were generalized by H. Weyl who essentially considered the coordinate functions in terms of other coordinates and to assume differentiability for the coordinate function, [1], [3], [4], [5].

2. Differential Manifolds

2.1. Basics

Definition 2.1: A topology on a set X is a collection T of subsets of X such that

ϕ and X are in T.

The union of an arbitrary collection of elements of T is in T.

The intersection of a finite collection of elements of T is in T, [6], [8].

Definition 2.2: A basis for a topology on a set X is a collection B of subsets of X such that

For each $x \in X$ there exists a $B \in \mathrm{B}$ containing x.

If $B_1, B_2 \in \mathrm{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \subset B_1 \cap B_2$ such that.

The basis B generates a topology T by defining a set $U \subset X$ to be open if for each $x \in U$ there exists a basis element $B \in \mathrm{B}$ with $x \in B \subset 0$, [6], [8].

Definition 2.3: Let X and Y be topological spaces. The product topology on the product set $X \times Y$ is generated by the basis elements $U \times V$, for all open sets $U \in X$ and $V \in Y$, [6], [9], [11].

Topology developed from the desire to generalize the notion of continuity of mappings of Euclidean spaces. That generalization is phrased as follows:

Definition 2.4: Let X and Y be topological spaces. A mapping $f: X \to Y$ is continuous if for each open set $U \subset Y$, the set $f^{-1}(U)$ is open in X, [6], [9], [11].

Definition 2.5: Let X and Y be topological spaces. A mapping $f: X \to Y$ is a homeomorphism if it is bijective and both f and f^{-1} are continuous. In this case X and Y are said to be homeomorphic.

When X and Y are homeomorphic, there is a bijective correspondence between both the points and the open sets of X and Y. Therefore, as topological spaces, X and Y are indistinguishable. This means that any property or theorem that holds for the space X that is based only on the topology of X also holds for Y, [6], [8].

Definition 2.6: A topological space X is said to be Hausdorff if for any two distinct points $x, y \in X$ there exist disjoint open sets U and V with $x \in U$ and $y \in V$, [6], [8].

Definition 2.7: A separation of a topological space X is a pair of disjoint open sets U, V such that $X = U \cup V$. If no separation of X exists, it is said to be connected, [6], [8].

2.2. Topological Manifolds

A manifold is a topological space that is locally equivalent to Euclidean space.

Definition 2.8: A manifold is a Hausdorff space *M* with a countable basis such that for each point $p \in M$ there is a neighborhood U of p that is homeomorphic to R^n for some integer n.

If the integer n is the same for every point in *M*, then M is called a n-dimensional manifold, [6], [8].

2.3. Differentiable Structures on Manifolds

Differentiation of mappings in Euclidean space is defined as a local property. Although a manifold is locally homeomorphic to Euclidean space, more structure is required to make differentiation possible. Any function on Euclidean space $f: R^n \to R$ is smooth or C^∞ if all of its partial derivatives exist.

A mapping of Euclidean spaces $f: R^m \to R^n$ can be thought of as a n-tuple of real-valued functions on $R^m, f = (f^1, \ldots\ldots, f^n)$, and f is smooth if each fi is smooth.

Given two neighborhoods U, V in a manifold *M*, two homeomorphisms $x: U \to R^n$ and $y: \mathrm{V} \to R^n$ are said to be C^∞-related if the mappings $x \circ y^{-1}: y(U \cap V) \to x(U \cap V)$ and $y \circ x^{-1}: x(U \cap V) \to y(U \cap V)$ are C^∞, [6], [8], [9], [11].

2.4. Smooth Functions and Mappings

The pair (x, U) is called a chart or coordinate system, and can be thought of as assigning a set of coordinates to points in the neighborhood U, figure 1. A collection of charts whose domains cover M is called an atlas, [5], [8], [9], [11].

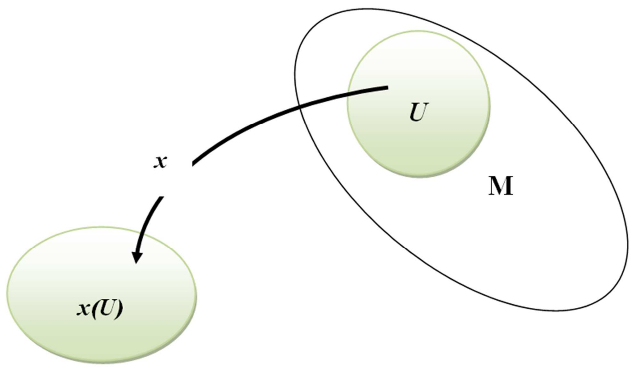

Figure 1. *A local coordinate system (x, U) on a manifold M.*

Definition 2.9: An atlas A on a manifold M is said to be maximal if for any compatible atlas A' on M any coordinate chart $(x, U) \in A'$ is also a member of A, [3], [6], [8].

Definition 2.10: A smooth structure on a manifold M is a maximal atlas A on M. The manifold M along with such an atlas is termed a smooth manifold.

Consider a function $f: M \to R$ on the smooth manifold M. This function is said to be a smooth function if for every coordinate chart (x,U) on M the function $f \circ x^{-1}: U \to R$ is smooth. More generally, a mapping $f: M \to N$ of smooth manifolds is said to be a smooth mapping if for each coordinate chart (x,U) on M and each coordinate chart (y,V) on N the mapping $y \circ f \circ x^{-1}: x(U) \to y(V)$ is a smooth mapping, [3], [6], [8].

Definition 2.11: Given two smooth manifolds M, N, a bijective mapping $f: M \to N$ is called a diffeomorphism if both f and f^{-1} are smooth mappings, [3], [6], [8], [10], [11].

2.5. Differentiable Manifolds

Differentiable manifolds are spaces that locally behave like Euclidean space. Much in the same way that topological spaces are natural for talking about continuity; differentiable manifolds are a natural setting for calculus. Notions such as differentiation, integration, vector fields, and differential equations make sense on differentiable manifolds. This section gives a review of the basic construction and properties of diferentiable manifolds, [3], [4], [8], [10].

Definition 2.12: Any atlas could be extended to maximal atlas by adding all charts that are C^∞ compatible with charts of A. The maximal atlas is called differentiable structure on the manifold M. A pair (M, A), for a topological manifold M of n-dimensions is called differential manifold, [3], [6], [8], [10].

3. Differentiation of Functions

There are various ways to define the derivative of a function on a differentiable manifold, the most fundamental of being the directional derivative. The definition of the directional derivative is complicated by the fact that a manifold will lack a suitable affine structure with which to define vector. The directional derivative therefore looks at curves in the manifold instead of vectors, [3], [6], [8].

3.1. Directional Differentiation

Given a real valued function f on an m dimensional differentiable manifold M, the directional derivative of f at a point p in M is defined as follows. Suppose that g (t) is a curve in M with $g(0) = p$, which is differentiable in the sense that its composition with any chart is a differential curve in R^n. Then the directional derivative of f at p along g is $df_{(g(t))}\big|_{t=0}$. The directional derivative only depends on the tangent vector of the curve at the point consindered, p for this case. Thus a definition of directional differentiation adapted to the cases of differentiable manifolds ultimately captures the intuitive features of directional differentiation in an affine space, [3], [6], [8], [11].

3.2. Tangent Spaces and Derivatives

Definition 3.1: Suppose that M is a smooth m-dimensional manifold of some Euclidean space R^n. Let $f: U \to M$ be a local parameterization around some point $x \in M$ with $\phi(0) = x$. The tangent space $T_x M$ is the image of the map $d\phi_0 : R^m \to R^n$ where $T_x M$ is an m-dimensional subspace of R^n. The vectors in this space are called tangent vectors.

Given a smooth map of manifolds M, N, $f: M \to N$, and a local parameterization $\phi: U \to M, \phi(0) = x \in M$ and $\phi: V \to N, \psi(0) = f(x) \in N$.

Let h be the map $h = \psi^{-1} \circ f \circ \phi: U \to V$, the we define the differential of f at x by $df_x : T_x M \to T_{f(x)} N$

$$df_x = d\psi_0 \circ dh_0 \circ d\phi_0^{-1}$$

The collection of tangent spaces at all points can in turn be made into a manifold, the tangent bundle, whose dimension is $2n$. The tangent bundle is where tangent vectors lie, and is itself a differentiable manifold, [3], [4], [5], [8], [10], [11].

3.3. Immersions, Submersions and Embeddings

The maps $df_x : T_x M \to T_{f(x)} N$ for all points $x \in M$ assemble to a map of tangent bundles,

Definition 3.2: A map f is called a submersion or we say that f is submersive if the linear map $df_x : T_x M \to T_{f(x)} N$ is an epimorphism (i.e., surjective) at each point x. It is called an immersion (or an immersive map) if the linear map $df_x : T_x M \to T_{f(x)} N$ is a monomorphism (i. e., injective) at each point. A smooth map $f: M \to N$ that is both injective and immersive is called embedding, [3], [4], [8], [10].

4. Conclusion

On a manifold that is sufficiently smooth, various kinds of jet bundles can also be considered. The tangent bundle of a manifold is the collection of curves in the manifold that is equivalent to the relation of first-order contact. Therefore, by analogy the k-th order tangent bundle is the collection of curves of the relation of k-th order contact. Likewise, the cotangent bundle is the bundle of one-jets of functions on the manifold: the k-jet bundle is the bundle of their k-jets. These and other examples of the general idea of jet bundles play a significant role in the study of differential operators on manifolds, [3], [5], [10].

It is worth noting that every topological manifold in n dimensions has a unique differential structure (up to diffeomorphism). Thus, the concepts of topological and differentiable manifold are only distinct in higher dimensions. It is known that in each higher dimension, there are some topological manifolds with no smooth structure, and some with multiple non-diffeomorphic structures, [2], [3], [6], [8].

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