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Complete Synchronization, Anti-synchronization and Hybrid Synchronization Between Two Different 4D Nonlinear Dynamical Systems

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Abstract: Three important phenomena of chaos synchronization are considered in this paper, In detailed, complete synchronization, anti- synchronization and hybrid synchronization based on the nonlinear active control approach between two different (non-identical) 4D hyperchaotic systems, i. e. Modified Pan and Liu are study herein. The Modified hyperchaotic Pan system is taken as drive system and hyperchaotic Liu system as response. Stabilization of error dynamics for each phenomenon is realized by satisfying two analytical approaches; Lyapunov's second method and linear system theory. Controllers are designed by using the relevant variable of drive and response systems. Theoretical analysis and numerical simulations are shown to verify the results.

Keywords: Chaos Synchronization, Complete Synchronization, Anti-synchronization, Hybrid Synchronization Nonlinear Dynamical Systems, Nonlinear Active Control

1. Introduction

In the last two decades, extensive studies have been done on the properties of nonlinear dynamical systems. One of the most important properties of nonlinear dynamical systems is that of chaos [1]. As a very interesting nonlinear phenomenon, chaos has been intensively investigated with the mathematics, engineering, science, and secure communication [2]. This phenomenon was firstly discovered by American meteorologist Edward N. Lorenz (1917-2008) when he studied a model of the Earth's atmospheric convection in 1963 [3-6].

Chaos control is one of the chaos phenomena, which contains two aspects, namely, chaos control and chaos synchronization. Chaos control and chaos synchronization were once believed to be impossible until the 1990s when Ott et al. developed the OGY method to suppress chaos, Pecora and Carroll introduced a method to synchronize two identical chaotic systems with different initial conditions [7-10], which opens the way for chaotic systems synchronization and a various techniques such as adaptive control, active control, nonlinear control, sliding mod control, back-stepping design method and so on have been successfully applied to chaos control and synchronization [11-13].

Chaos control and chaos synchronization play very important role in the study of nonlinear dynamical systems and have great significance in the application of chaos [14]. Especially, the subject of chaos synchronization has received considerable attention due to its potential applications in physics, secure communication, chemical reactor, biological networks, control theory, artificial neural network. etc. [15].

Various types of synchronization phenomena have been presented such as complete synchronization (CS), generalized synchronization (GS), lag synchronization, anti-synchronization (AS), projective synchronization (PS), generalized projective synchronization (GPS) [16]. The most familiar synchronization phenomena are complete synchronization and anti- synchronized [17, 18].

Complete synchronization is characterized by the equality of state variables evolving in the time, while anti-synchronization is characterized by the disappearance of sum of relevant variables evolving in the time. Projective synchronization is characterized by the fact that the drive and response systems could be synchronized up to a scaling factor, whereas in generalized projective synchronization, the responses of the synchronized dynamical states synchronize up to a constant scaling matrix α . It is easy to see that the complete synchronization and anti- synchronization are special cases of generalized projective synchronization where the scaling matrix $\alpha = I$ and $= -I$ respectively [19].

In the hybrid synchronization scheme, one part of the system is synchronized and the other part is anti-synchronized so that the complete synchronization and anti- synchronization coexist in the system [20].

In this paper, we discuss some important phenomena for chaos synchronization i.e. complete synchronization, anti-synchronization and hybrid synchronization between two different 4D hyperchaotic systems via nonlinear active control, The results derived in this paper are established using the Lyapunov's second method and linear system theory.

2. Description of Hyperchaotic Modified Pan and Liu Systems

According to the Ref [21], the modified hyperchaotic Pan system which described by the following dynamical system

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = cx_1 - x_1x_3 + x_4 \\ \dot{x}_3 = x_1x_2 - bx_3 \\ \dot{x}_4 = -\,dx_2 \end{cases} \tag{1}$$

where x_1, x_2, x_3, x_4 are the state variables and $a, b, c, d > 0$ are the parameters of the system. When parameters $a = 10, b = 8/3, c = 28$ and $d = 10$, system (1) is hyperchaotic and has two positive Lyapunov exponents, i.e. $LE_1 = 0.24784, LE_2 = 0.08194$, and has only one equilibrium $O(0,0,0,0)$. This equilibrium is an unstable under these parameters.

In 2006, Wang et al. [22], presented the four-dimensional hyperchaotic Liu system which described by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = rx_1 - kx_1x_3 + x_4 \\ \dot{x}_3 = hx_1^2 - px_3 \\ \dot{x}_4 = -\,qx_1 \end{cases} \tag{2}$$

where $\mathrm{a, r, k, h, p, q} > 0$ are system parameters. When parameters $\mathrm{a} = 10, \mathrm{p} = 2.5, \mathrm{r} = 40, \mathrm{q} = 10.6, \mathrm{k} = 1$ and $\mathrm{h} = 4$, system (2) is hyperchaotic and has only equilibrium $\mathrm{O}(0,0,0,0)$, and the equilibrium is an unstable saddle node under these parameters. Figure 1 and Figure 2 shows the attractors of the system (1) and the system (2) respectively.

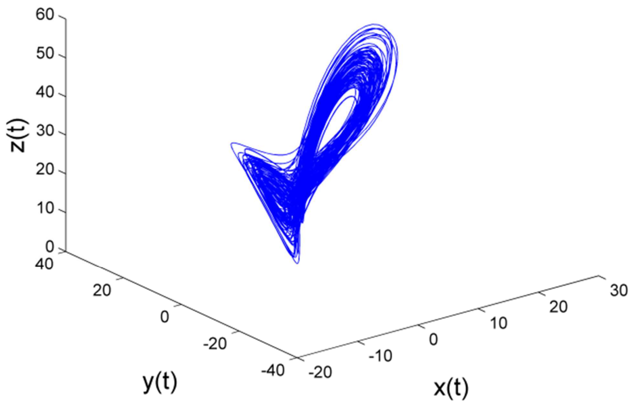

Fig. 1. *The attractor of system (1) in x-y-z space.*

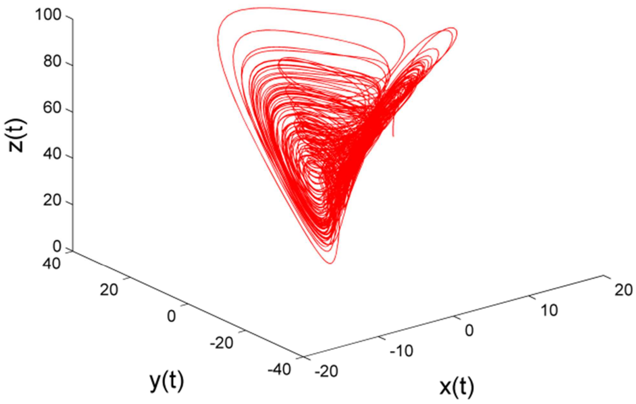

Fig. 2. *The attractor of system (2) in x-y-z space.*

3. Complete Synchronization Between Modified Pan and Liu Systems

In this section, the synchronization behavior between two different hyperchaotic systems is achieved via nonlinear active control strategy. In order to observe synchronization between hyperchaotic Modified Pan system and hyperchaotic Liu system, we consider the system (1) as the drive system and hyperchaotic Liu system as the response system which describe by the following system

$$\begin{cases} \dot{y}_1 = a(y_2 - y_1) + u_1 \\ \dot{y}_2 = ry_1 - ky_1y_3 + y_4 + u_2 \\ \dot{y}_3 = hy_1^2 - py_3 + u_3 \\ \dot{y}_4 = -\,qy_1 + u_4 \end{cases} \tag{3}$$

where $u = [u_1, u_2, u_3, u_4]^T$ is the controller to be designed. The synchronization error $e \epsilon R^4$ is defined as:

$$e_i = y_i - \alpha_i x_i\,;\ i = 1,2,3,4, \forall \alpha_i = 1 \tag{4}$$

where α_i is a scaling factor taken the value 1 for synchronization and -1 for anti-synchronization according to the projective synchronization approach. So, subtracting the above system from the system (1), we get the error dynamical system between the drive system and the response system which is given by:

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1 \\ \dot{e}_2 = re_1 + e_4 + (r-c)x_1 - ky_1y_3 + x_1x_3 + u_2 \\ \dot{e}_3 = -pe_3 + (b-p)x_3 + hy_1^2 - x_1x_2 + u_3 \\ \dot{e}_4 = -qe_1 - de_2 - qx_1 + dy_2 + u_4 \end{cases} \tag{5}$$

We need to find the nonlinear active control law for $u_i, \forall i$ in such a manner that the error dynamics of (5) is globally asymptotically stable.

Choosing the control functions $u = [u_1, u_2, u_3, u_4]^T$ as

$$\begin{cases} u_1 = v_1 \\ u_2 = (c-r)x_1 + ky_1y_3 - x_1x_3 + v_2 \\ u_3 = (p-b)x_3 - hy_1^2 + x_1x_2 + v_3 \\ u_4 = qx_1 - dy_2 + v_4 \end{cases} \tag{6}$$

Substituting Eq.(6) in Eq.(5) gives

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + v_1 \\ \dot{e}_2 = re_1 + e_4 + v_2 \\ \dot{e}_3 = -pe_3 + v_3 \\ \dot{e}_4 = -qe_1 - de_2 + v_4 \end{cases} \tag{7}$$

Where $v = [v_1, v_2, v_3, v_4]^T$ are the linear control inputs chosen such that the system (7) becomes stable.

Let us consider

$$[v_1, v_2, v_3, v_4]^T = A[e_1, e_2, e_3, e_4]^T \tag{8}$$

where A is a 4×4 constant matrix. In order to make above system stable, the matrix A should be selected in such a way that all of its eigenvalues with negative real parts. Consider the following choice of matrix A as

$$\begin{bmatrix} 0 & -a & 0 & 0 \\ -r & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ q & d & 0 & -1 \end{bmatrix} \tag{9}$$

With this choice, the error system (5) become

$$\begin{cases} \dot{e}_1 = -ae_1 \\ \dot{e}_2 = -e_2 \\ \dot{e}_3 = -pe_3 \\ \dot{e}_4 = -e_4 \end{cases} \tag{10}$$

Hence, we arrive at the following results.

Theorem 1. System (1) will complete synchronize with the system (3) if nonlinear active control is chosen as

$$\begin{cases} u_1 = -ae_2 \\ u_2 = -re_1 - e_2 - e_4 + (c-r)x_1 + ky_1y_3 - x_1x_3 \\ u_3 = (p-b)x_3 - hy_1^2 + x_1x_2 \\ u_4 = qe_1 + de_2 - e_4 + qx_1 - dy_2 \end{cases} \tag{11}$$

Proof. Based on the Lyapunov second method, we construct a positive definite Lyapunov candidate function as

$$V(e) = e^TPe = \tfrac{1}{2}e_1^2 + \tfrac{1}{2}e_2^2 + \tfrac{1}{2}e_3^2 + \tfrac{1}{2}e_4^2 \tag{12}$$

where $P = diag[1,1,1,1]$, The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\dot{V}(e) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$

$$= e_1(-ae_1) + e_2(-e_2) + e_3(-pe_3) + e_4(-e_4)\ \dot{V}(e) = -ae_1^2 - e_2^2 - pe_3^2 - e_4^2 = -e^TQe \tag{13}$$

Where

$$Q = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

According to Ref [21], every diagonal matrix with positive diagonal elements are positive definite. So $Q > 0$. Therefore, $\dot{V}(e)$ is negative definite. And according to the Lyapunov asymptotical stability theory, the nonlinear active controller is achieved and the synchronization of the hyperchaotic systems is achieved. The proof is now complete.

Numerically, we justified these analytical results by MATLAB program and take the initial values of the drive system and the response system are $(10,15,20,30)$ and $(-10,-5,0,5)$ respectively. Figure 3 and Figure 4 show the synchronization between systems with nonlinear active control (11)

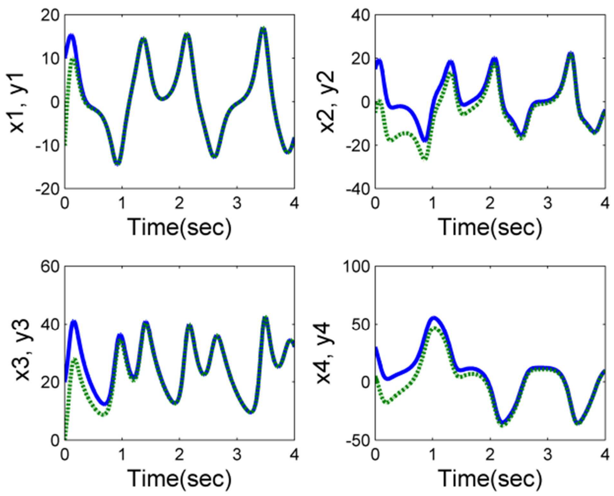

Fig. 3. *Synchronization between systems (1) and (3) with nonlinear active control (11).*

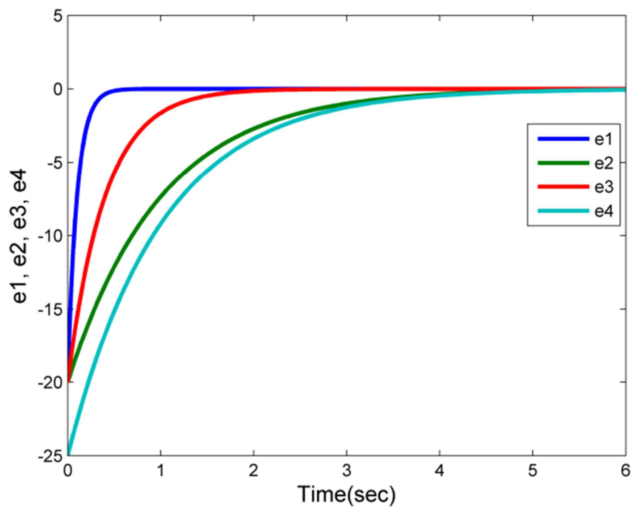

Fig. 4. *The converges of system (5) with controllers (11).*

Remark 1. Controller (11) is constructing from combine the linear control inputs v in Eq. 9 with Eq. 6.

Remark 2. We can proof Theorem 1 by a second method where the linear system (10) can be written as

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = \begin{bmatrix}-a & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -p & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} \quad (14)$$

Then the eigenvalues are $\lambda_1 = -a$, $\lambda_{2,3} = -1, \lambda_4 = -p$. So, all roots with negative real parts. Hence, in light of the linear system theory, the error dynamical system (5) is asymptotically stable with the controller (11). Consequently, the synchronization of the drive system (1) and the response system (3) is achieved. So, nonlinear active control based on two analytical methods; Lyapunov second method and the linear system theory as a main tool.

4. Anti-Synchronization Between Modified Pan and Liu Systems

In this section, we discuss anti-synchronization behavior of non-identical Modified hyperchaotic Pan system and hyperchaotic Liu system. Consider Modified hyperchaotic Pan as drive system and hyperchaotic Liu system as a response system (system3). We define the error states for anti- synchronization as

$$e_i = y_i - \alpha_i x_i \; ; \; i = 1,2,3,4, \forall \alpha_i = -1 \quad (15)$$

The corresponding error dynamics system obtained by adding Eq. 1 and Eq.3 is

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) + u_1 \\ \dot{e}_2 = re_1 + e_4 + (c-r)x_1 - ky_1y_3 - x_1x_3 + u_2 \\ \dot{e}_3 = -pe_3 + (p-b)x_3 + hy_1^2 + x_1x_2 + u_3 \\ \dot{e}_4 = -qe_1 - de_2 + qx_1 + dy_2 + u_4 \end{cases} \quad (16)$$

Theorem 2. System (1) will anti-synchronize with system (3) if nonlinear active control is chosen as

$$\begin{cases} u_1 = (a-1)e_1 - ae_2 \\ u_2 = -re_1 - e_2 - e_4 + (r-c)x_1 + ky_1y_3 + x_1x_3 \\ u_3 = (p-1)e_3 + (b-p)x_3 - hy_1^2 - x_1x_2 \\ u_4 = qe_1 + de_2 - e_4 - qx_1 - dy_2 \end{cases} \quad (17)$$

Proof. Substituting the controllers (17) in the error dynamical system (16) we have

$$\begin{bmatrix}\dot{e}_1\\ \dot{e}_2\\ \dot{e}_3\\ \dot{e}_4\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\end{bmatrix} \quad (18)$$

According to the previous discussion, there are two method.

In Lyapunov second method, consider the Lyapunov function as

$$V(e) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{1}{2}e_3^2 + \frac{1}{2}e_4^2$$

The derivative of the Lyapunov function $V(e)$ with respect to time is

$$\dot{V}(e) = e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_4\dot{e}_4$$

$$= e_1(-e_1) + e_2(-e_2) + e_3(-e_3) + e_4(-e_4)\, \dot{V}(e) = -e_1^2 - e_2^2 - e_3^2 - e_4^2 = -e^TQe$$

where $Q = diag[1,1,1,1]$. So, Q is appositive definite and $\dot{V}(e)$ is negative definite. The proof is now complete based on Lyapunov method. In the linear system theory, the characteristic values of the matrix of the system (18) has all roots with negative real part $\lambda_{1,2,3,4} = -1$. Which implies that the system (1) anti- synchronize the system (3). The proof is complete based on the linear system theory. Figure 5 and Figure 6 show verify these results numerically.

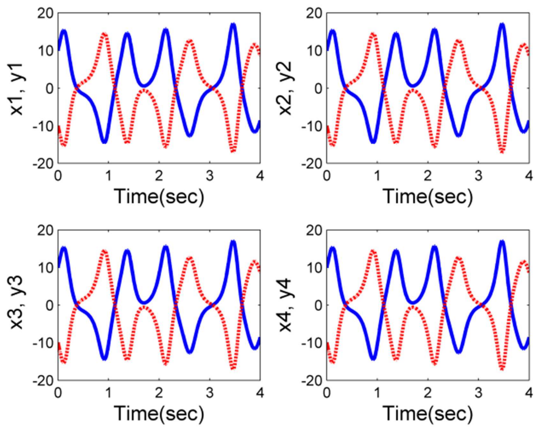

Fig. 5. *Anti-synchronization between systems (1) and (3) with active control (17).*

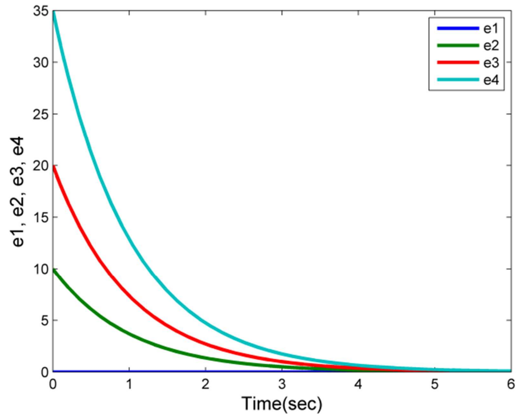

Fig. 6. *The converges of system (16) with controllers (17).*

5. Hybrid Synchronization Between Modified Pan and Liu Systems

Hybrid synchronization based on the nonlinear active control between two different 4D hyperchaotic systems, i.e. Modified Pan and Liu systems is consider in this section.

Modified hyperchaotic Pan system is taken as drive (system 1) and hyperchaotic Liu system as response (system 3). Control is designed by using the relevant variables of drive and response systems.

The hybrid synchronization error is defined by

$$\begin{cases} e_1 = y_1 - x_1 \\ e_2 = y_2 + x_2 \\ e_3 = y_3 - x_3 \\ e_4 = y_4 + x_4 \end{cases} \tag{19}$$

System (19) can be written in a succinct form as

$$e_i = \begin{cases} y_i - x_i, if\ i\ is\ odd \\ y_i + x_i, if\ i\ is\ even \end{cases} \tag{20}$$

The error dynamics is easily obtained as

$$\begin{cases} \dot{e}_1 = a(e_2 - e_1) - 2ax_2 + u_1 \\ \dot{e}_2 = re_1 + e_4 + (r+c)x_1 - ky_1y_3 - x_1x_3 + u_2 \\ \dot{e}_3 = -pe_3 + (b-p)x_3 + hy_1^2 - x_1x_2 + u_3 \\ \dot{e}_4 = -qe_1 - de_2 - qx_1 + dy_2 + u_4 \end{cases} \tag{21}$$

i. e. subtracting and adding the system (3) from the system (1) respectively.

Theorem 3. System (1) will hybrid synchronize with the system (3) if nonlinear active control is chosen as

$$\begin{cases} u_1 = -ae_2 + 2ax_2 \\ u_2 = -re_1 - e_2 - e_4 - (r+c)x_1 + ky_1y_3 + x_1x_3 \\ u_3 = (p-1)e_3 + (p-b)x_3 - hy_1^2 + x_1x_2 \\ u_4 = qe_1 + de_2 - e_4 + qx_1 - dy_2 \end{cases} \tag{22}$$

Proof. Rewrite system (21) with control (22) as follows

$$\begin{cases} \dot{e}_1 = -ae_1 \\ \dot{e}_2 = -e_2 \\ \dot{e}_3 = -e_3 \\ \dot{e}_4 = -e_4 \end{cases} \tag{23}$$

It is clear that all roots with negative real part $\lambda_1 = -a, \lambda_{2,3,4} = -1$, which implies that the system (1) hybrid synchronize the system(3) based on the linear system theory under controller (22). Figure 7 and Figure 8 show verify these results numerically.

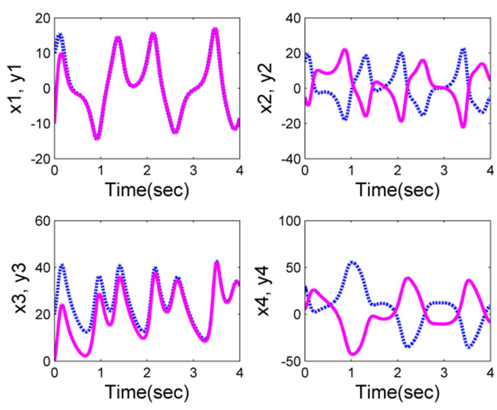

Fig. 7. *Hybrid synchronization between systems (1) and (3) with active control (22).*

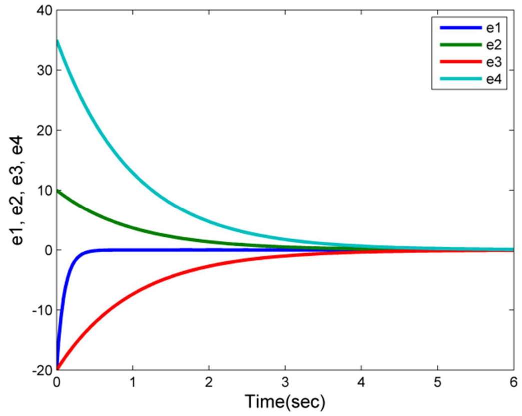

Fig. 8. *The converges of system (21) with controllers (22).*

6. Conclusions

In this paper deals with chaos synchronization between two non-identical hypechaotic systems through nonlinear active control technique. And succeed to achieve three important subjects which include complete synchronization, anti- synchronization and hybrid synchronization based on Lyapunov's second method and linear system theory. Numerical simulations are used to verify the effectiveness of the proposed control.

We believe that the results of this research work should be beneficial and could be employed to increase contribution to the scientific literature on the methods of chaos synchronization which may have applications in different fields of engineering including secure communication, hybrid image encryption, genetic networks.

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