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Minimum Time Problem for *n×n* Co-operative Hyperbolic Lag Systems

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Abstract: In this paper, a minimum time problem for $n\times n$ co-operative hyperbolic systems involving Laplace operator and with time-delay is considered. First, the existence of a unique solution of such hyperbolic system with time-delay is proved. Then necessary conditions of a minimum time control are derived in the form of maximum principle. Finally the bang-bang principle and the approximate controllability conditions are investigated.

Keywords: Time-Optimal Control Problem, Co-operative Systems, Hyperbolic Systems with Time Delay, Approximate Controllability, Bang-Bang Principle

1. Introduction

The most widely studies of the problems in the mathematical theory of control are the "time optimal" problems. The simple version is that steering the initial state y_0 in a Hilbert space H to hitting a target set $K\subset H$ in minimum time , with control subject to constraints ($u\in U\subset H$). In this paper we will focus our attention on some special aspects of minimum time problems for co-operative hyperbolic systems with time delay. In order to explain the results we have in mind, it is convenient to consider the abstract form: Let V and H be two real Hilbert spaces such that V is a dense subspace of H .Identifying the dual of H with H we may consider $V\subset H\subset V'$. Let $A(t)$ ($t\in]0,T[$) be a family of continuous operators associated with a bilinear forms $\pi(t;.,.)$ defined on $V\times V$ which are symmetric and coercive on V. Then ,from Lions [1] and Lions - Magenes [2]and for B be a bounded linear operator from a Hilbert space U to $L^2(0,T;H)$, the following abstract systems:

$$\left.\begin{aligned} &\frac{d^2}{dt^2}y(t)+A(t)y(t)=Bu(t), \quad t\in]0,T[,\\ &y(0)=y_0, \quad y_0\in V,\\ &y'(0)=y_1, \quad y_1\in H.\end{aligned}\right\} \tag{1}$$

have a unique weak solution y such that $(y,y')\in C([0,T];V\times H)$. We shall denote by $y(t;u)$ the unique solution of the equation (1) corresponding to the control u. The time optimal control problem we shall concern reads:

$$Min\{\tau : y(\tau;u)\in K, u\in U\} \tag{2}$$

where K is a given subset of H, which is called the target set of the Problem (2). A control u^0 is called a time optimal control if $u^0\in U$ and if there exists a number $\tau^0>0$ such that $y(\tau^0;u^0)\in K$ and

$$\tau^0=\min\{\tau : y(\tau;u)\in K, u\in U\} \tag{3}$$

We call the number τ^0 as the optimal time for the time optimal control Problem (3).

Three questions (problems) arise naturally in connection with this problem

(a) Does there exist a control u, and $\tau > 0$ such that $y(\tau;u)\in K$? (this is an approximate controllability problem).

(b) Assume that the answer to (a) is in the affirmative, does there exist a control u^0 which steering $y(\tau^0;u^0)$ to hitting a target set K in minimum time?

(c) If u^0 exists,is it unique? what additional properties does it have?

In practical applications, the behavior of many dynamical systems which describes a state of time-optimal control problems depends upon their past histories. This phenomenon can be induced by the presence of time delays. Due to the inherent difficulties in solving control problems with time delays, the progress in this area has been slow. Here, we mention the work of Wang [4], where the time optimal control for a class of ordinary differential-difference equation with time lag was considered. Also, we mention the work of Knowles [5], where a Time optimal control of parabolic systems with boundary condition involving time delays was considered and it is shown that the optimal control is characterized in terms of an adjoint system and it is of the bang-bang type.

Time-optimal control of distributed parameter systems governed by a system of hyperbolic equations is of special importance for the active control of structural systems for which the equations of motion are generally expressed by hyperbolic differential equations. A typical application of a hyperbolic equation is the vibrating system.Time-optimal control of distributed parameter systems governed by a system of hyperbolic equations have been studied in many papers, we mention only [6], [7] in which time optimal distributed control problems of vibrating systems has been studied. In our papers [8-11], the results in [6] and [7] have been extended to the time optimal control problems for systems governed by $n\times n$ hyperbolic systems, involving laplace operator with different cases of observations.

In this paper, we will consider a time-optimal control problem for the following $n\times n$ co-operative linear hyperbolic system with time delay h and involving Laplace operator (here and everywhere below the vectors are denoted by bold letters.):

$$\left.\begin{array}{ll}
\frac{\partial^2 y_i}{\partial t^2}(x,t)-(A(t)\mathbf{y})_i = d_i y_i(x,t-h)+u_i(x,t) & \text{in } \Omega\times]0,T[,\\
y_i(x,t)=\phi_i(x,t) & \text{in } \Omega\times]-h,0[,\\
y_i(x,0)=y_{i,0}(x) & \text{in } \Omega,\\
\frac{\partial y_i}{\partial t}(x,0)=y_{i,1}(x) & \text{in } \Omega,\\
y_i(x,t)=0, & \text{on } \Sigma=\Gamma\times]-h,T[
\end{array}\right\} \quad (4)$$

where $\phi_i, y_{i,0}, y_{i,1}$ are given functions, $\Omega\subset R^N$ is a bounded open domain with smooth boundary Γ and $A(t)$ ($t\in]0,T[$) are a family of $n\times n$ continuous matrix operators,

$$A(t)y=\begin{pmatrix}\Delta+a_1 & a_{12} & & a_{1n}\\ a_{21} & \Delta+a_2 & & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & & \Delta+a_n\end{pmatrix}\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{pmatrix}$$

with co-operative coefficient functions d_i, a_i, a_{ij} satisfying the following conditions:

$$\left.\begin{array}{l}
d_i, a_i, a_{ij} \text{ are positive functions in } L^\infty(Q),\\
a_{ij}=a_{ji} \text{ (symmetry conditions)},\\
a_{ij}(x,t)\le\sqrt{a_i(x,t)a_j(x,t)}.
\end{array}\right\} \quad (5)$$

This problem is, steering the initial vector state $\mathbf{y}(0)$ for system (4), with a vector control function $\mathbf{u}=(u_1,u_2,...,u_n)$ belonging to a given control set U_ε^n so that an observation $\mathbf{y}(t)$ hitting a given target set K_ε^n in minimum time,

$$\begin{array}{l}
U_\varepsilon^n=\{\phi=(\phi_1,\phi_2,...\phi_n)\in(L^2(Q))^n : \|\phi_i\|_{L^2(Q)}\le\varepsilon\}\\
K_\varepsilon^n=\{z=(z_1,z_2,...z_n)\in(L^2(\Omega))^n : \|z_i-z_{id}\|_{L^2(\Omega)}\le\varepsilon\},
\end{array} \quad (6)$$

and $\varepsilon,\varepsilon>0$ and $z_{id}\in L^2(\Omega)$ are given.

First, we establish the well posedness of the system (4) under conditions on the coefficients stated by the principal eigenvalue of the Laplace eigenvalue problem. Then, we formulate a time optimal control problem and we derive the necessary and sufficient conditions which the optimal control must satisfy in terms of the adjoint.

2. Solutions of $n\times n$ Co-operative Hyperbolic Systems

Let $H_0^1(\Omega)$, be the usual Sobolev space of order one which consists of all $\phi\in L^2(\Omega)$ whose distributional derivatives $\frac{\partial\phi}{\partial x_i}\in L^2(\Omega)$ and $\phi_\Gamma=0$ with the scalar product norm

$$<y,\phi>_{H_0^1(\Omega)}=<y,\phi>_{L^2(\Omega)}+<\nabla y,\nabla\phi>_{L^2(\Omega)},$$

$$\text{where } \nabla = \sum_{k=1}^{N} \frac{\partial}{\partial x_k}.$$

We have the following dense embedding chain [13]

$$(H_0^1(\Omega))^n \subseteq (L^2(\Omega))^n \subseteq (H_0^{-1}(\Omega))^n.$$

where $H_0^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

For $\mathbf{y}=(y_i)_{i=1}^n, \phi=(\phi_i)_{i=1}^n \in (H_0^1(\Omega))^n$ and $t\in]0,T[$, let us define a family of continues bilinear forms

$$\pi(t;.,.): (H_0^1(\Omega))^n \times (H_0^1(\Omega))^n \to \Re \text{ by}$$

$$\begin{aligned}\pi(t;\mathbf{y},\phi) &= \sum_{i=1}^{n}\int_\Omega \left[(\nabla y_i)(\nabla \phi_i) - a_i(x,t) y_i \phi_i\right] dx \\ &\quad -2\sum_{i>j}^{n}\int_\Omega a_{ij}(x,t) y_j \phi_i dx \\ &= \sum_{i=1}^{n}\int_\Omega \left[(-\Delta y_i) - a_i(x,t) y_i\right]\phi_i dx \\ &\quad -2\sum_{i>j}^{n}\int_\Omega a_{ij}(x,t) y_j \phi_i dx \\ &= \sum_{i=1}^{n} < -(A(t)\mathbf{y})_i, \phi >_{L^2(\Omega)}\end{aligned} \tag{7}$$

Lemma 1 If Ω is a regular bounded domain in R^N, with boundary Γ, and if m is positive on Ω and smooth enough (in particular $m\in L^\infty(\Omega)$,) then the eigenvalue problem:

$$\left.\begin{aligned} -\Delta y &= \lambda m(x) y && \text{in } \Omega, \\ y &= 0 && \text{on } \Gamma \end{aligned}\right\}$$

possesses an infinite sequence of positive eigenvalues:

$$0<\lambda_1(m)<\lambda_2(m)\le \ldots \lambda_k(m)\ldots; \lambda_k(m)\to\infty, \text{as} k\to\infty.$$

Moreover $\lambda_1(m)$ is simple, its associate eigenfunction e_m is positive, and $\lambda_1(m)$ is characterized by:

$$\lambda_1(m)\int_\Omega m y^2 dx \le \int_\Omega |\nabla y|^2 dx \tag{8}$$

Proof. See[14].

Now, let

$$\lambda_1(a_i)\ge n, \quad i=1,2,......,n \tag{9}$$

Lemma 2 *If (5) and (9) hold then, the bilinear form (7) satisfy the Gårding inequality*

$$\pi(t;\mathbf{y},\mathbf{y}) + c_0 \| y \|^2_{\left(L^2(\Omega)\right)^n} \ge c_1 \| y \|^2_{\left(H_0^1(\Omega)\right)^n}, \quad c_0, c_1 > 0$$

Proof. In fact

$$\begin{aligned}\pi(t;\mathbf{y},\mathbf{y}) &= \sum_{i=1}^{n}\int_\Omega \left[|\nabla y_i|^2 - a_i(x,t) y_i^2\right] dx \\ &\quad -\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x,t) y_i y_j dx \\ &\ge \sum_{i=1}^{n}\int_\Omega \left[|\nabla y_i|^2 - a_i(x,t) y_i^2\right] dx \\ &\quad -2\sum_{i>j}^{n}\int_\Omega \sqrt{a_i(x,t) a_j(x,t)}\, y_i y_j dx\end{aligned}$$

By Cauchy Schwarz inequality and (8),we obtain

$$\begin{aligned}\pi(t;\mathbf{y},\mathbf{y}) \ge & \sum_{i=1}^{n}\left(1-\frac{1}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx \\ & -2\sum_{i>j}^{n}\int_\Omega \left(\frac{1}{\sqrt{\lambda_1(a_i)\lambda_1(a_j)}}\right) \\ & \left(\int_\Omega |\nabla y_i|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla y_j|^2 dx\right)^{\frac{1}{2}} \\ \ge & \sum_{i=1}^{n}\left(\frac{\lambda_1(a_i)-n}{\lambda_1(a_i)}\right)\int_\Omega |\nabla y_i|^2 dx\end{aligned}$$

From (9) we have

$$\pi(t;\mathbf{y},\mathbf{y}) \ge \alpha\left[\sum_{i=1}^{n}\int_\Omega |\nabla y_i|^2 dx\right] \quad \alpha>0$$

Add $\| \mathbf{y} \|_{(L^2(\Omega))^n}$ to two sides, then we have the result.

For simplicity, We introduce the following notations: for $j=0,1,....,$ let $I^j = [(j-1)h, jh[,$ $Q^j = \Omega\times I^j$ and

$$W(I) = \left\{\begin{aligned} &\phi:\phi\in L^2(I;(H_0^1(\Omega))^n), \quad \frac{\partial\phi}{\partial t}\in L^2(I;(L^2(\Omega))^n), \\ &\frac{\partial^2\phi}{\partial t^2}\in L^2(I;(H_0^{-1}(\Omega))^n) \end{aligned}\right\}.$$

For optimal control problems it is of importance to consider the cases where the control u_i belongs to $L^2(Q)$. For these cases, we have the following results:

Theorem 1 Let (5), (9) be hold and let $y_{i,0}, y_{i,1}\, \phi_i\, u_i$ be given with

$$y_{i,0}\in H^1(\Omega), y_{i,1}\in L^2(\Omega), \phi_i\in W(I^0), u_i\in L^2(Q)$$

Then there exist a unique solution $\mathbf{y}\in W(0,T)$ satisfying the Dirichlet problem (4). Moreover $y_i(.,jh)\in H_0^1(\Omega)$ and $\frac{\partial y_i}{\partial t}(.,jh)\in L^2(\Omega)$ for $j=1,2,...$ and $i=1,2,...,n.$

The above theorem can be established by first solving the problem on Q^1 then, the existence of a unique solution on Q^2 is established by using the solution on Q^1 to generate the initial data at $t=h$. This advancing process is repeated for $Q^3,Q^4,\ldots$ etc until the procedure covers the whole cylinder $Q.$ Hereafter, the solution on Q^j will be denoted by $\mathbf{y}^j\ j=1,2,\ldots.$ Now, the existence of a unique solution $\mathbf{y}^j\in W(I^j)\subset(L^2(Q))^n$ can be established by making use of results of Lions [1] (Theorem 1.1 chapter4)specialized to the case of $V=(H_0^1(\Omega))^n$, $H=(L^2(\Omega))^n$ and an initial data at $t=(j-1)h.$ In order to apply the same results of Lions to any Q^j, we must verify that the right hand sides of (4)on Q^j, satisfy the same conditions as required for $y_{i,0},y_{i,1}$ and u_i this means, we must verify that

$$y_i^{j-1}(x,(j-1)h)\in H_0^1(\Omega),\quad \frac{\partial y_i^{j-1}}{\partial t}(x,(j-1)h)\in L^2(\Omega)$$

This can be shown by making use of [1] (Remark.1.3 Chapter 4).

3. Co-operative Hyperbolic Systems

We will denote by $\mathbf{y}(t;\mathbf{u})$ to the unique solution of (4), at time t corresponding to a given control $\mathbf{u}\in U_\varepsilon^n$ and a given functions $y_{i,0},y_{i,1}\,\phi_i\,u_i$ satisfying the hypothesis of Theorem 1. Occasionally, we write $\mathbf{y}(x,t;\mathbf{u})$ when the explicit dependence on x is required.

In this section, we consider the following first time-optimal control problem with control $\mathbf{v}$ acts in velocity initial condition and position observation $\mathbf{y}(x,t;\mathbf{v})$:

$$(TOP):\quad \min\{t:\mathbf{y}(x,t;\mathbf{u})\in K_\varepsilon^n,\mathbf{u}\in U_\varepsilon^n\},$$

In order for the problem (TOP) to be well posed, we assume the following

There exist a $\tau\in]0,T]$ and $\mathbf{u}\in U_\varepsilon^n$ with

$$\mathbf{y}(\tau;\mathbf{u})\in K_\varepsilon^n \tag{10}$$

Set

$$\tau_1^0=\inf\{\tau:\mathbf{y}(\tau;\mathbf{v})\in K_\varepsilon^n \text{ foresome } \mathbf{v}\in U_\varepsilon^n\}. \tag{11}$$

The following result holds .

Theorem 2 *If (5), (9) are hold, then there exist an admissible control* $\mathbf{u}^0$ *to the problem (TOP), which steering* $\mathbf{y}(t;\mathbf{u}^0)$ *to hitting a target set* K_ε^n *in minimum time* τ^0 *(defined by (11)). Moreover*

$$\sum_{i=1}^n\int_\Omega (y_i(\tau^0;\mathbf{u}^0)-z_{id})\big(y_i(\tau^0;\mathbf{u})-y_i(\tau^0;\mathbf{u}^0)\big)dx\ge 0 \qquad \forall\mathbf{u}\in U_\varepsilon^n \tag{12}$$

Proof. Fixe x, we can choose $\tau^m\to\tau^0$ and admissible controls $\{\mathbf{u}^m\}$ such that

$$\mathbf{y}(\tau^m;\mathbf{u}^m)\in K_\varepsilon^n,\quad m=1,2,\ldots.$$

Set $\mathbf{y}^m=\mathbf{y}(\mathbf{u}^m).$ Since U_ε^n is bounded, we may verify that $\mathbf{y}^m$ (respectively $\frac{d\mathbf{y}}{dt}$)ranges in a bounded set in $(L^2(0,T;(H_0^1(\Omega))^n))$ (respectively $(L^2(0,T;(L^2(\Omega))^n)=(L^2(Q))^n)$.

We may then extract a subsequence, again denoted by $\{\mathbf{u}^m,\mathbf{y}^m\}$ such that

$$\left.\begin{array}{l}\mathbf{u}^m\to\mathbf{u}^0\quad \text{weaklyin}(L^2(Q))^n,\quad u^0\in U_\varepsilon^n,\\ \mathbf{y}^m\to\mathbf{y}\quad \text{weaklyin}L^2\left(0,T;\left(H_0^1(\Omega)\right)^n\right)\\ \dfrac{d\mathbf{y}^m}{dt}\to\quad \text{weaklyin}(L^2(Q))^n\end{array}\right\} \tag{13}$$

We deduce from the equality

$$\frac{d^2\mathbf{y}^m}{dt^2}=\mathbf{u}^m-A(t)\mathbf{y}^m$$

that

$$\frac{d^2\mathbf{y}^m}{dt^2}\to\frac{d^2\mathbf{y}}{dt^2}=\mathbf{u}^0-A(t)\mathbf{y}\quad \text{in}L^2\left(0,T;\left(H^{-1}(\Omega)\right)^n\right),$$

and

$$\mathbf{y}(0)=\mathbf{y}_0\quad \frac{d\mathbf{y}}{dt}(0)=\mathbf{y}_1.$$

But

$$\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\begin{array}{l}=\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^m)\\ +\mathbf{y}(\tau^0;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\end{array}$$

then, from (13) we have

$$\mathbf{y}(\tau^0;\mathbf{u}^m)\to\mathbf{y}(\tau^0;\mathbf{u}^0)\quad \text{weaklyin}(H_0^1(\Omega))^n \tag{14}$$

and

$$\begin{aligned}&\|\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^m)\|_{\left(L^2(\Omega)\right)^n}\\ &=\left\|\int_{\tau^0}^{\tau^m}\frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt\right\|_{\left(L^2(\Omega)\right)^n}\\ &\le\sqrt{\tau^m-\tau^0}\left(\int_{\tau^0}^{\tau^m}\left\|\frac{d}{dt}\mathbf{y}(t;\mathbf{u}^m)dt\right\|^2_{\left(L^2(\Omega)\right)^n}dt\right)^{\frac{1}{2}}\\ &\le c\sqrt{\tau_n-\tau^0}\end{aligned} \tag{15}$$

Combine(14) and (15)show that

$$\mathbf{y}(\tau^m;\mathbf{u}^m)-\mathbf{y}(\tau^0;\mathbf{u}^0)\to 0\quad \text{weaklyin}\left(L^2(\Omega)\right)^n. \tag{16}$$

Similarly, we can verify that

$$\mathbf{y}'(\tau^m;\mathbf{u}^m)-\mathbf{y}'(\tau^0;\mathbf{u}^0)\to 0\quad \text{weaklyin}\left(H_0^{-1}(\Omega)\right)^n. \tag{17}$$

and so, $\mathbf{y}(\tau^0;\mathbf{u}^0)\in K_\varepsilon^n$ as K_ε^n is closed and convex, hence weakly closed. This shows that K_ε^n is reached in time τ^0 by admissible control $\mathbf{u}^0$.

For the second part of the theorem, really, from Theorem 1, the mapping $t\to\mathbf{y}(t;\mathbf{u})$ and $t\to\mathbf{y}'(t;\mathbf{u})$ from $[0,T]\to(H_0^1(\Omega))^n$ and $[0,T]\to(L^2(\Omega))^n$ respectively are continuous for each fixed $\mathbf{u}$ and so $\mathbf{y}(\tau^0;\mathbf{u})\notin int K_\varepsilon^n$, for any $\mathbf{u}\in U_\varepsilon^n$, by minimality of τ^0.

Using Theorem 1 it is easy to verify that the mapping $\mathbf{u}\to\mathbf{y}(\tau^0;\mathbf{u})$, from $(L^2(Q))^n\to(L^2(\Omega))^n$, is continuous and linear. then, the set

$$\mathsf{A}(\tau^0)=\{\mathbf{y}(\tau^0;\mathbf{u}):\mathbf{u}\in U_\varepsilon^n\}$$

is the image under a linear mapping of a convex set hence $\mathsf{A}(\tau^0)$ is convex. Thus we have $\mathsf{A}(\tau^0)\cap int K_\varepsilon^n=\varnothing$ and $\mathbf{y}(\tau^0;\mathbf{u}^0)\in\partial K_\varepsilon^n$ (boundary of K_ε^n) . Since $int K_\varepsilon^n\ne\varnothing(from(10))$ so there exists a closed hyperplane separating $A(\tau^0)$ and K_ε^n containing $\mathbf{y}(\tau^0;\mathbf{u}^0)$, i.e there is a nonzero $\mathbf{g}\in(L^2(\Omega))^n$ such as

$$\begin{aligned}&\sup_{\mathbf{y}\in A(\tau^0)}<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u})>_{(L^2(\Omega))^n}\le<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u}^0)>\\ &_{(L^2(\Omega))^n}\le\inf_{\mathbf{y}\in K_\varepsilon^n}<\mathbf{g},\mathbf{y}(\tau^0;\mathbf{u})>_{(L^2(\Omega))^n}\end{aligned} \tag{18}$$

From the second inequality in (18), $\mathbf{g}$ must support the set K_ε^n at $\mathbf{y}(\tau^0;\mathbf{u}^0)$ i.e

$$<\mathbf{g},(\mathbf{y}(\tau^0;\mathbf{u})-\mathbf{y}(\tau^0;\mathbf{u}^0))>_{(L^2(\Omega))^n}\ge 0\quad\forall\mathbf{u}\in U_\varepsilon^n$$

and since $(L^2(\Omega))^n$ is a Hilbert space, $\mathbf{g}$ must be of the form

$$\mathbf{g}=\lambda(\mathbf{y}(\tau^0;\mathbf{u}^0)-z_{id})\quad \text{forsome}\,\lambda>0.$$

Dividing the inequality (18) by λ gives the desired result.

The above condition (12) can be simplified by introducing the following adjoint equation. For each $\mathbf{u}^0\in U_\varepsilon^n$, we define $p(x,t;\mathbf{u}^0)$ as the solution of the following system

$$\left.\begin{array}{l}\dfrac{\partial^2 p_i}{\partial t^2}(t;\mathbf{u}^0)-\left(A(t)\mathbf{p}(t;\mathbf{u}^0)\right)_i\\ =\begin{cases}p_i(x,t+h;\mathbf{u}), & \text{in}\,\Omega\times]0,\tau^0-h[\\ \quad 0\quad \text{in}\,\Omega\times]\tau^0-h,\tau^0[\end{cases}\\ p_i(x,\tau^0;\mathbf{u}^0)=0\quad \text{in}\,\Omega,\\ \dfrac{\partial p_i}{\partial t}(x,\tau^0;\mathbf{u}^0)=-(y_i(x,\tau^0;\mathbf{u}^0)-z_{id})\quad \text{in}\,\Omega,\\ p_i(x,t;\mathbf{u}^0)=0\quad \text{in}\,\Gamma\times]0,\tau^0[.\end{array}\right\} \tag{19}$$

For simplicity, we introduce the following notations: $I_{\tau^0}^j=[\tau^0-jh,\tau^0-(j-1)h[$, $Q_{\tau^0}^j=\Omega\times I_{\tau^0}^j$. We observe that for given z_{id} and u_i, Problem (19) can be solved backward in time starting from $t=\tau^0$ by first obtaining the solution $\mathbf{p}=\mathbf{p}^1$ on $Q_{\tau^0}^1$ i.e we solve

$$\left.\begin{array}{l}\dfrac{\partial^2 p_i^1}{\partial t^2}+(A(t)\mathbf{p}^1)_i=0\quad \text{in}\,Q_{\tau^0}^1\\ p_i^1(x,\tau^0;\mathbf{u}^0)=0\quad \text{in}\,\Omega,\\ \dfrac{\partial p_i^1}{\partial t}(x,\tau^0;\mathbf{u}^0)=-(y_i^0(x,\tau^0)-z_{id})\quad \text{in}\,\Omega,\\ p_i^1(x,t)=0\quad \text{on}\,]0,\tau^0[\times\Gamma.\end{array}\right\} \tag{20}$$

Having found $\mathbf{p}^1$ we may proceed to solve the problem on $Q^2_{\tau^0}$ backward in time starting with terminal data at $t=\tau^0-h$: i.e we solve

$$\left.\begin{aligned}
&\frac{\partial^2 p_i^2}{\partial t^2}+(A(t)\mathbf{p}^2)_i=0 \quad \text{in } Q^2_{\tau^0}\\
&p_i^2(x,\tau^0-h;\mathbf{u}^0)=0 \quad \text{in } \Omega,\\
&\frac{\partial p_i^2}{\partial t}(x,\tau^0;\mathbf{u}^0)=\frac{\partial p_i^1}{\partial t}(x,\tau^0-h;\mathbf{u}) \quad \text{in } \Omega,\\
&p_i^1(x,t)=0 \quad \text{on }]0,\tau^0[\times\Gamma.
\end{aligned}\right\} \tag{21}$$

Note that the right-hand sides of (21) are completely determined once $\mathbf{p}^1$ is Known. This backward process is repeated until the procedure covers the whole cylinder Q. For $\mathbf{u}\in(L^2(Q))^n$, the existence of a unique solution $\mathbf{p}^1\in W(I^1_{\tau^0})$ (with $p_i^1(.,\tau^0)\in H^1(\Omega)$, and $\frac{\partial p_i^1}{\partial t}(.,\tau^0)\in L^2(\Omega)$,) can be established by applying Theorem 1 to (21) with obvious change of variables and with reversed sense of time $t'=\tau^0-t$. In the same way, the result can be extended to $Q^j, j=2,3,...$ Thus, we have the result:

Lemma 3 Let the hypothesis of Theorem.1 be satisfied. Then, for given z_{id} in $L^2(\Omega)$ and any $u_i\in L^2(Q)$, there exists a unique solution $\mathbf{p}(t,\mathbf{u})\in W(0,\tau^0)$ to Problem (19).

Now, in view of Lemma.3, we can proceed to simplify Inequality (12) using the adjoint System (19). Multiply the first equation in (19) by $y_i(\mathbf{u})-y_i(\mathbf{u}^0)$ and integrate over $]0,\tau^0[\times\Omega$ we option the identity:

$$\begin{aligned}
&\int_\Omega (y_i(\tau^0;\mathbf{u}^0)-z_{id})(y_i(\tau^0;\mathbf{u})-y_i(\tau^0;\mathbf{u}^0)dx=\\
&=\int_0^{\tau^0}\int_\Omega\left[p_i(t)\left(\frac{\partial^2}{\partial t^2}(y_i(\mathbf{u})-y_i(\mathbf{u}^0))+\left(A(t)(\mathbf{y}(\mathbf{u})-\mathbf{y}(\mathbf{u}^0))\right)_i\right)\right]dxdt\\
&+\int_0^{\tau^0-\omega}\int_\Omega p_i(t+\omega)(y_i(\mathbf{u})-y_i(\mathbf{u}^0))dxdt
\end{aligned} \tag{22}$$

The first term in the right hand side of (22)can be rewrite as

$$\begin{aligned}
&\int_0^{\tau^0}\int_\Omega\left[p_i(t)\left(\frac{\partial^2}{\partial t^2}(y_i(\mathbf{u})-y_i(\mathbf{u}^0))+\left(A(t)(\mathbf{y}(\mathbf{u})-\mathbf{y}(\mathbf{u}^0))\right)_i\right)\right]dxdt\\
&=\int_0^{\tau^0}\int_\Omega p_i(t)(\mathbf{u}-\mathbf{u}^0)dxdt-\int_0^{\tau^0}\int_\Omega p_i(t)(y_i(t-h,\mathbf{u})-y_i(t-h,\mathbf{u}^0))dxdt\\
&=\int_0^{\tau^0}\int_\Omega p_i(t;\mathbf{u}^0)(u_i-u_i^0)dxdt-\int_0^{\tau^0-\omega}\int_\Omega p_i(t+\omega)(y_i(\mathbf{u})-y_i(\mathbf{u}^0))dxdt
\end{aligned} \tag{23}$$

Substituting from (23) into (22) we obtain

$$\begin{aligned}
&\int_\Omega (y_i(\tau^0;\mathbf{u}^0)-z_{id})(y_i(\tau^0;\mathbf{u})-y_i(\tau^0;\mathbf{u}^0)dx\\
&=\int_0^{\tau^0}\int_\Omega p_i(t;\mathbf{u}^0)(u_i-u_i^0)dxdt
\end{aligned}$$

Hence (12) is equivalent to

$$\sum_{i=1}^n\int_0^{\tau^0}\int_\Omega p_i(t;\mathbf{u}^0)(u_i-u_i^0)dxdt\ge 0 \quad \forall\mathbf{u}\in U^n_\varepsilon. \tag{24}$$

We now summarize the foregoing result

Theorem 3 *We assume that (5), (9) hold. Then there exist the adjoint state*

$$\mathbf{p}\in\left\{\mathbf{p}:\mathbf{p}\in L^2(0,\tau^0;(H^1_0(\Omega))^n),\quad \frac{\partial\mathbf{p}}{\partial t}\in(L^2(\Omega))^n)\right\}$$

such that the optimal control $\mathbf{u^0}$ of problem (TOP) is characterized by (19), (24) together with (4) (with $u_i=u_i^0, i=1,2,...,n$).

4. Bang-Bang Control and Controllability

The maximum conditions (24) of the optimal control leads to the following result:

Theorem 4 *(Bang-Bang theorem) Let the hypothesis of Theorem.3 be satisfied. Then the optimal control of (TOP) is unique and bang -bang, that is* $|u_i^0(x,t)|\equiv 1$ *almost everywhere.*

Proof. The proof will follow from Theorem3 if we can show that $p_i(x,t)\neq 0$ for almost all $(x,t)\in\Omega\times]0,\tau^0[$. We shall show this fact by contradiction. Therefore, we suppose that

$$p_i(x,t)=0 \quad \text{for} \quad (x,t)\in E\subset\Omega\times]0,\tau^0[,$$

E non null. Let us denote by k_0 the largest nonnegative integer k such that $\tau^0-kh>0$.

Suppose firstly that E has non null intersection with $\Omega\times]\tau^0-h,\tau^0[$. If $\tau^0<h$, the same argument applies to $\Omega\times(0,\tau^0)$. In the cylinder $\Omega\times]\tau^0-h,\tau^0[$, $\mathbf{p}(\mathbf{u}^0)$ satisfies

$$\frac{\partial^2 p_i(\mathbf{u}^0)}{\partial t^2}+(A(t)\mathbf{p}(\mathbf{u}^0))_i=0,(x,t)\in\Omega\times]\tau^0-h,\tau^0[$$

and so, by ([?]), $p_i(\mathbf{u}^0)$ must be analytic in the cylinder $\Omega\times]\tau^0-h,\tau^0[$. As $p_i(\mathbf{u}^0)$ is zero in E, it must be identically zero in $\overline{\Omega}\times]\tau^0-h,\tau^0[$. From our earlier remarks, the mapping $t \to \dfrac{\partial p_i}{\partial t}(t;\mathbf{u}^0)$ is continuous from $[0,T] \to L^2(\Omega)$, and so

$$\frac{\partial p_i}{\partial t}(\tau^0;\mathbf{u}^0) = 0 = -(y_i(\tau^0;\mathbf{u}^0) - z_i),$$

which leads to a contradiction.

We shall now show that the case where $E \cap \Omega \times]0, \tau^0 - kh[$ is non null can be inductively reduced to the above.

Note that, in the cylinder $\overline{\Omega} \times]\tau^0 - 2h, \tau^0 - h[$, $p_i(\mathbf{u}^0)$ satisfies

$$\frac{\partial^2 p_i(\mathbf{u}^0)}{\partial t^2} + (A(t)\mathbf{p}(\mathbf{u}^0))_i = p_i(x, t+h;\mathbf{u}^0), \qquad (x,t) \in \Omega \times]\tau^0 - 2h, \tau^0 - h[. \tag{25}$$

We have just shown that, $p_i|_\Omega (x, t+h;\mathbf{u}^0)$ is analytic for $(x,t) \in \Omega \times]\tau^0 - 2h, \tau^0 - h[$ and so $p_i(\mathbf{u}^0)$ must be analytic in $\overline{\Omega} \times]\tau^0 - 2h, \tau^0 - h[$, since (25) have analytic coefficients [?]. By induction, $p_i(\mathbf{u}^0)$ must be analytic in each cylinder

$$\overline{\Omega} \times]\tau^0 - kh, \tau^0 - (k-1)h[, \quad k = 2,3,\ldots,k_0$$
$$,\text{and } \overline{\Omega} \times]0, \tau^0 - k_0 h[.$$

If $p_i(\mathbf{u}^0)) = 0$ on $E \cap \Omega \times]\tau^0 - kh, \tau^0 - (k-1)h[$ for some $k = 2,3,\ldots,k_0$, then by analyticity and continuity as before,

$$p_i(\mathbf{u}^0)) \equiv 0 \quad \text{for} \quad (x,t) \in \overline{\Omega} \times]\tau^0 - kh, \tau^0 - (k-1)h[. \tag{26}$$

Substituting (26) in to (25) gives

$$\frac{\partial^2 p_i(\mathbf{u}^0)}{\partial t^2} + (A(t)\mathbf{p}(\mathbf{u}^0))_i = 0 \quad \text{for}$$
$$(x,t) \in \Omega \times]\tau^0 - (k-1)h, \tau^0 - (k-2)h[.$$

So in the cylinder $\overline{\Omega} \times]\tau^0 - (k-1)h, \tau^0 - (k-2)h[$, $p_i(\mathbf{u}^0)$ satisfies

$$\frac{\partial^2 p_i(\mathbf{u}^0)}{\partial t^2} + (A(t)\mathbf{p}(\mathbf{u}^0))_i = 0,$$
$$p_i(., \tau^0 - (k-1)h;\mathbf{u}^0)) = 0;$$

consequently, by backward uniqueness [12],

$$p_i(\mathbf{u}^0) \equiv 0, \quad \overline{\Omega} \times]\tau^0 - (k-1)h, \tau^0 - (k-2)h[.$$

We can repeat this argument until $p_i(\tau^0;\mathbf{u}^0) = 0,$ which leads to a contradiction. Since U_ε^n is strictly convex, then the optimal control is unique. This is complete the proof.

With regard to controllability assumption (10), We can show quit easily that (4) is approximately controllable in $(L^2(Q))^n$ in any finite time $\tau > 0,$ if and only if., $\{\mathbf{y}(\tau;\mathbf{u}) : \mathbf{u} \in (L^2(Q))^n\}$ is dense in $(L^2(Q)^n$. By the Hahn-Banach theorem, this will be the case if

$$\int_\Omega \bar{z}_i(x) y_i(x,\tau;\mathbf{u})dx = 0, \quad \bar{z}_i \in L^2(\Omega), \tag{27}$$

for all $\mathbf{u} \in (L^2(Q))^n$, implies that $\bar{z}_i(x) = 0, i = 1,2,\ldots,n.$

Let $\mathbf{p} \in W(0,\tau)$ be the unique solution of (19) with

$$p_i(x,\tau) = \bar{z}_i(x), \quad x \in \Omega.$$

The proof of Theorem 3 showed that

$$\int_\Omega \bar{z}_i(x)(y_i(t;\mathbf{u}) - y_i(t;\bar{\mathbf{u}})dx = \int_0^\tau \int_\Omega p_i(t)(u_i - \bar{u}_i))dxdt;$$

and so, if (27)holds for all $\mathbf{u} \in (L^2(Q))^n$,

Then

$$\int^\tau \int_\Omega p_i u_i dxdt = 0,$$

$\mathbf{u} \in (L^2(Q))^n$ and $\mathbf{p} = 0$ in Q. By continuity,

$$p_i(x,\tau) = \bar{z}_i(x) = 0$$

for almost all $x \in \Omega$.

5. Scalar Cases

In this section, we give some special cases.

Case I: Coupled system with time delay

Here, we take the case where $n = 2,$ in this case, the time optimal problem therefore is

$$\min\{t : \mathbf{y}(x,t;\mathbf{u}) \in K_\varepsilon^2, \mathbf{u} = (u_1,u_2) \in U_\varepsilon^2\}$$

The state $\mathbf{y}=(y_1,y_2)$ is the solution of the following equations

$$\left.\begin{array}{ll}
\dfrac{\partial^2 y_1}{\partial t^2}-\Delta y_1=a_{11}(x,t)y_1+a_{12}(x,t)y_2+y_1(x,t-h)+u_1^0, & x\in\Omega,\quad t\in]0,\tau^0[,\\
\dfrac{\partial^2 y_2}{\partial t^2}-\Delta y_2=a_{21}(x,t)y_1+a_{22}(x,t)y_2+y_2(x,t-h)+u_2^0, & x\in\Omega,\quad t\in]0,\tau^0[,\\
y_1(x,t)=\phi_1(x,t),\quad y_2(x,t)=\phi_2(x,t) & x\in\Omega,\quad t\in]-h,0[,\\
y_1(x,0)=y_{1,0}(x),\quad y_2(x,0)=y_{2,0}(x), & x\in\Omega,\\
\dfrac{\partial y_1}{\partial t}(x,0)=y_{1,1}(x),\quad \dfrac{\partial y_1}{\partial t}(x,0)=y_{2,1}(x), & x\in\Omega,\\
y_1(x,t)=y_2(x,t)=0, & x\in\Gamma,\quad t\in]0,\tau^0[,
\end{array}\right\}$$

With

$$\left.\begin{array}{l}
a_{ij}(x,t),i,j=1,2 \text{ arepositivefunctionsin } L^\infty(Q),\\
\lambda_1(a_{11})\ge 2,\quad \lambda_1(a_{22})\ge 2,\quad a_{12}=a_{21},\\
\phi_1,\phi_2\in W(I^0).
\end{array}\right\}$$

The adjoint $\mathbf{p}=(p_1,p_2)$ is the solution of the following equations

$$\left.\begin{array}{l}
\dfrac{\partial^2 p_1}{\partial t^2}(t;\mathbf{u}^0)-\Delta p_1=a_{11}(x,t)p_1+a_{12}(x,t)p_2+\begin{cases}p_1(x,t+h;\mathbf{u}^0), & \text{in}\,\Omega\times]0,\tau^0-h[\\ 0 & \text{in}\,\Omega\times]\tau^0-h,\tau^0[\end{cases}\\
\dfrac{\partial^2 p_2}{\partial t^2}(t;\mathbf{u}^0)-\Delta p_2=a_{21}(x,t)p_1+a_{22}(x,t)p_2+\begin{cases}p_2(x,t+h;\mathbf{u}^0), & \text{in}\,\Omega\times]0,\tau^0-h[\\ 0 & \text{in}\,\Omega\times]\tau^0-h,\tau^0[\end{cases}\\
p_1(x,\tau^0;\mathbf{u}^0)=p_2(x,\tau^0;\mathbf{u}^0)=0\quad \text{in}\,\Omega,\\
\dfrac{\partial p_1}{\partial t}(x,\tau^0;\mathbf{u}^0)=-(y_1(x,\tau^0;\mathbf{u}^0)-z_{1d})\quad \text{in}\,\Omega,\\
\dfrac{\partial p_2}{\partial t}(x,\tau^0;\mathbf{u}^0)=-(y_2(x,\tau^0;\mathbf{u}^0)-z_{2d})\quad \text{in}\,\Omega,\\
p_1(x,t;\mathbf{u}^0)=p_2(x,t;\mathbf{u}^0)=0\quad \text{in}\,\Gamma\times]0,\tau^0[.
\end{array}\right\}$$

The maximum condition is

$$\int_0^{\tau^0}\int_\Omega\left[p_1(x,0;u^0)(u_1-u_1{}^0)+p_2(x,0;u^0)(u_1-u_1{}^0)\right]dxdt\ge 0\quad \forall u\in U_\varepsilon^2.$$

Case II: $n=1$ with time delay

Here, we take the case where $n=1$, in this case, the time optimal problem therefore is

$$\min\{t: y(x,t;u)\in K_\varepsilon^1, u=u_1\in U_\varepsilon^1\}$$

The state $y=y_1$ is the solution of the following equations

$$\left.\begin{array}{ll}\dfrac{\partial^2 y}{\partial t^2} - \Delta y = a_1(x,t)y + y(x,t-h) + u^0, & x\in\Omega,\quad t\in]0,\tau^0[,\\ y(x,t) = \phi(x,t), & x\in\Omega,\quad t\in]-h,0[,\\ y(x,0) = y_0(x),\quad \dfrac{\partial y}{\partial t}(x,0) = y_1(x), & x\in\Omega,\\ y(x,t) = 0, & x\in\Gamma,\quad t\in]0,\tau^0[,\end{array}\right\}$$

with

$$\left.\begin{array}{l} a_1(x,t), \text{ispositivefunctionin } L^\infty(Q),\\ \lambda_1(a_1)\ge 1,\quad \phi\in W(I^0).\end{array}\right\}$$

The adjoint $p = p_1$ is the solution of the following equations

$$\left.\begin{array}{l}\dfrac{\partial^2 p}{\partial t^2}(t;u^0) - \Delta p = a_1(x,t)p + \begin{cases} p(x,t+h;u^0), & \text{in}\Omega\times]0,\tau^0-h[\\ 0 & \text{in}\Omega\times]\tau^0-h,\tau^0[\end{cases}\\ p(x,\tau^0;u^0) = 0 \quad \text{in}\Omega,\\ \dfrac{\partial p}{\partial t}(x,\tau^0;u^0) = -(y(x,\tau^0;u^0) - z_d) \quad \text{in}\Omega,\\ p_1(x,t;u^0) = 0 \quad \text{in}\Gamma\times]0,\tau^0[.\end{array}\right\}$$

The maximum condition is

$$\int_0^{\tau^0}\int_\Omega \left[p(x,0;u^0)(u-u^0)\right]dx\,dt \ge 0 \quad \forall u\in U_\varepsilon^1.$$

Case III: $n=1$ without delay

Here, we take the case where $n=1$, without time delay (see [1], [7]) in this case:

The state $y = y_1$ is the solution of the following equations

$$\left.\begin{array}{ll}\dfrac{\partial^2 y}{\partial t^2} - \Delta y = a_1(x,t)y + u^0, & x\in\Omega,\quad t\in]0,\tau^0[,\\ y(x,0) = y_0(x),\quad \dfrac{\partial y}{\partial t}(x,0) = y_1(x), & x\in\Omega,\\ y(x,t) = 0, & x\in\Gamma,\quad t\in]0,\tau^0[,\end{array}\right\}$$

with

$$\left.\begin{array}{l} a_1(x,t), \text{ispositivefunctionin } L^\infty(Q),\\ \lambda_1(a_1)\ge 1.\end{array}\right\}$$

The adjoint $p = p_1$ is the solution of the following equations

$$\left.\begin{array}{ll}\dfrac{\partial^2 p}{\partial t^2}(t;u^0) - \Delta p = a_1(x,t)p & x\in\Omega,\quad t\in]0,\tau^0[,\\ p(x,\tau^0;u^0) = 0 & x\in\Omega,\\ \dfrac{\partial p}{\partial t}(x,\tau^0;u^0) = -(y(x,\tau^0;u^0) - z_d) & x\in\Omega,\\ p_1(x,t;u^0) = 0 & x\in\Gamma,\quad t\in]0,\tau^0[.\end{array}\right\}$$

The maximum condition is

$$\int_0^{\tau^0}\int_\Omega \left[p(x,0;u^0)(u-u^0)\right]dx\,dt \ge 0 \quad \forall u \in U_\varepsilon^1.\cdot$$

6. Comments

- We note that, in this paper, we have chosen to treat a special systems involving Laplace operator, just for simplicity. Most of the results we described in this paper apply, without any change on the results, to more general parabolic systems involving the following second order operator :

$$L(x,.) = \sum_{i,j=1}^{n} b_{ij}(x,.)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,.)\frac{\partial}{\partial x_j} + b_0(x,.)$$

with sufficiently smooth coefficients (in particular, $b_{ij}, b_j, b_0 \in L^\infty(Q), b_j, b_0 > 0$) and under the Legendre-Hadamard ellipticity condition

$$\sum_{i,j=1}^{n} \eta_i \eta_j \ge \sigma \sum_{i=1}^{n} \eta_i \quad \forall (x,t) \in Q,$$

for all $\eta_i \in \Re$ and some constant $\sigma > 0$.

In this case, we replace the first eigenvalue of the Laplace operator by the first eigenvalue of the operator L (see [9]).

- In this paper, we have chosen to treat a co-operative hyperbolic systems with Dirichlet boundary onditions.The results can be extended to the case of $n \times n$ co-operative hyperbolic system with Neumann boundary conditions: If we take $H^1(\Omega)$ instead of $H_0^1(\Omega)$, we have to replace the Dirichlet boundary conditions $y_i = 0, p_i = 0$ on the boundary by Neumann boundary conditions $\frac{\partial y_i}{\partial \nu} = 0, \frac{\partial p_i}{\partial \nu} = 0$ where ν is the outward normal.

- In this paper, we have taken a simple target set K_ε^n. In (TOP) (for example), if we take

$$K_\varepsilon^n = \{z \in (L^2(\Omega))^n : \| z_i - z_{id} \|_{L^2(\Omega)} + \sum_{j=1}^{N} \| \frac{\partial z_i}{\partial x_j} - z_{id} \|_{L^2(\Omega)} \le \varepsilon\}$$

then the necessary optimality conditions coincide with (19),(24), (4) (with $v_i = v_i^0, i = 1,2$) and $(y_i(x,\tau_1^0;\mathbf{v}^0) - z_{id})$ in (19) is replaced by $(-\Delta_x + I)(y_i(x,\tau_1^0;\mathbf{v}^0) - z_{id})$. Also, we can take another observation (see [7]).

- The results in this paper, carry over to the optimal control problems with fixed -time ([1] chapter 4), for example, he results of (TOP) carry over to the fixed -time problem

$$\text{minimize} \sum_{i=1}^{n} \int_\Omega | y_i(x,T;u) - z_{id}(x) |^2 \, dx, \quad T \text{ fixed},$$

subject to (4) [except in the trivial case where $z_{id}(x) = y_i(x,T;\mathbf{v})$ for some admissible control $\mathbf{v} = (v_i)_{i=1}^n$.] This can proven in an analogous manner, as the necessary and sufficient conditions for optimality for this problem coincide with (19), (24) and (4) (with $v_i = v_i^0, i = 1,2,...,n$).

- As a final coment, we note that the control problem for the second order evolution system (4) can be reduced to a similar control problem first order system; in the usual way: set $\psi = \begin{pmatrix} \mathbf{y} \\ \frac{\partial \mathbf{y}}{\partial t} \end{pmatrix}$ and rewrite (4) in the first order form. However, the existing results on the time-optimal problem ([1], [10], [11]) pertain to the case where the observation is only one case (position-velocity) but here we can take different cases.

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