Mathematics Letters
2017; 3(4): 40-45
http://www.sciencepublishinggroup.com/j/ml
doi: 10.11648/j.ml.20170304.11

Some Induced Averaging Aggregation Operators Based on Pythagorean Fuzzy Numbers

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To cite this article:
Khaista Rahman, Saleem Abdullah, Asad Ali, Fazli Amin. Some Induced Averaging Aggregation Operators Based on Pythagorean Fuzzy Numbers. *Mathematics Letters.* Vol. 3, No. 4, 2017, pp. 40-45. doi: 10.11648/j.ml.20170304.11

Received: May 29, 2017; **Accepted**: August 10, 2017; **Published**: September 6, 2017

Abstract: In this paper we present two new types aggregation operators such as, induced Pythagorean fuzzy ordered weighted averaging aggregation operator and induced Pythagorean fuzzy hybrid averaging aggregation operator. We also discuss of important properties of these proposed operators and construct some examples to develop these operators.

Keywords: Pythagorean Fuzzy Sets, I-PFOWA Operator, I-PFHA Operator

1. Introduction

Atanassov [1] introduced the concept of IFS characterized by a membership function and a non-membership function. It is more suitable for dealing with fuzziness and uncertainty than the ordinary fuzzy set developed by Zadeh [2] characterized by one membership function. Gau and Buehrer [3] proposed the notion of vague set. Chen and Tan [4] and Hong and Choi [5] presented some techniques for handling multi-criteria fuzzy decision-making problems based on vague sets. Bustine and Burillo [6] showed that the vague set is equivalent to IFS. In 1986, many scholars [7, 8, 9, 10, 11, 12] have done works in the field of AIFS and its applications. Particularly, information aggregation is a very crucial research area in IFS theory that has been receiving more and more focus. Xu [13] developed some basic arithmetic aggregation operators, including IFWA operator, IFOWA operator and IFHA operator and applied them to group decision making. Xu and Yager [14] defined some basic geometric aggregation operators such as, IFWG operator, IFOWG operator and IFHG operator, and applied them to multiple attribute decision making (MADM) based on intuitionistic fuzzy information. Wang and Liu [15] introduced the notion of IFEWG operator and geometric IFEOWG operator and applied them to group decision making. In [16] Wang and Liu also introduced the concept of IFEWA operator and IFEOWA operator. Zhao and Wei [17] introduced the notion of two new types of hybrid aggregation operators such as, IFEHA operator and IFEHG operator. But there are many cases where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that their sum is greater than one. Therefore, Yager [18] introduced the concept of PFS. PFS is more powerful tool to solve uncertain problems. In 2013, Yager and Abbasov [19] introduced the notion of two new Pythagorean fuzzy aggregation operators such as PFWA operator and PFOWA operator. In [20, 21, 22, 23] K. Rahman et al. introduced the concept of PFHA operator, PFWG operator, PFOWG operator and PFHG operator.

Thus keeping the advantages of the above mention aggregation operators in this article we introduce the notion of two new types aggregation operators based on PFNs, such as, induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator and induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator. We also discuss some of their basic properties including idempotency, boundedness, commutativity and monotonicity. We also give some examples to develop these proposed operators.

The remainder paper can be constructed as. In Section 2, we present some basic definitions which will be used in our later sections. In Section 3, we introduce the notion of induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator and induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator. In Section 4, we have conclusion.

2. Preliminaries

Definition 1: [16, 17] Let Z be a fixed set, then Pythagorean fuzzy set can be defined as:

$$P=\left\{\left\langle z,\mu_P(z),\eta_P(z)\right\rangle \mid z\in Z\right\}, \tag{1}$$

where $\mu_P(z)$ and $\eta_P(z)$ are mappings from Z to [0,1], such that $0\le\mu_P(z)\le 1,$ $0\le\eta_P(z)\le 1,$ and $0\le\mu_P^2(z)+\eta_P^2(z)\le 1,$ for all $z\in Z$, and represent the degrees of membership and nonmembership of the component $z\in Z$ to set P. Let

$$\pi_P(z)=\sqrt{1-\mu_P^2(z)-\eta_P^2(z)}, \tag{2}$$

then it is called the Pythagorean fuzzy index of the component $z\in Z$ to set P, and represents the degree of indeterminacy of z to P. Also $0\le\pi_P(z)\le 1$ for every $z\in Z.$

Definition 2: [31] Let $\alpha=(t_\alpha,1-f_\alpha)$ be a PFN, then the score function and the accuracy function of α can be defined as:

$$S(\alpha)=t_\alpha^2-f_\alpha^2, \tag{3}$$

and

$$H(\alpha)=t_\alpha^2+f_\alpha^2, \tag{4}$$

where $S(\alpha)\in[-1,1]$ and $H(\alpha)\in[0,1].$

If α and β be two Pythagorean fuzzy numbers, then we have

(1) If $S(\alpha)<S(\beta),$ then $\alpha<\beta$

(2) If $S(\alpha)=S(\beta),$ then

(a) If $H(\alpha)=H(\beta),$ then $\alpha=\beta,$

(b) If $H(\alpha)<H(\beta)$ then $\alpha<\beta,$

(c) If $H(\alpha)>H(\beta)$ then $\alpha>\beta.$

3. Induced Pythagorean Fuzzy Averaging Aggregation Operators

In this section, we introduce the notion of two induced Pythagorean fuzzy aggregation operators such as, induced Pythagorean fuzzy ordered weighted averaging aggregation operator and induced Pythagorean fuzzy hybrid averaging aggregation operator. We also discuss some of their basic properties and construct some examples to develop these proposed operators.

3.1. Induced Pythagorean Fuzzy Ordered Weighted Averaging Operator

Definition 3: An induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator can be define as:

$$\text{I-PFOWA}_\omega\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\dots,\langle u_n,\alpha_n\rangle\right)=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right), \tag{5}$$

where $\omega=(\omega_1,\omega_2,\dots,\omega_n)^T$ is the weighted vector of $\alpha_j\,(j=1,2,\dots,n)$ with conditions $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1.$ $\alpha_{\sigma(j)}$ is the α_j value of the PFOWA pair $\langle u_j,\alpha_j\rangle$ having the j^{th} largest u_j and u_j in $\langle u_j,\alpha_j\rangle$ is referred to as the order inducing variable and α_j as the Pythagorean fuzzy argument variable

Theorem 1: Let $\langle u_j,\alpha_j\rangle(j=1,2,\dots,n)$ be a collection of 2-tuples, then their aggregated value by using the I-PFOWA operator is also PFV, and

$$\text{I-PFOWA}_\omega\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\dots,\langle u_n,\alpha_n\rangle\right)=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right), \tag{6}$$

where $\omega=(\omega_1,\omega_2,\dots,\omega_n)^T$ is the weighted vector of $\alpha_j\,(j=1,2,3,\dots,n)$ with $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1.$

Proof: We can prove this theorem by mathematical induction. First we show that equation (6) holds for $n=2.$ Since

$$\omega_1\alpha_{\sigma(1)}=\left(\sqrt{1-\left(1-t^2_{\alpha_{\sigma(1)}}\right)^{\omega_1}},1-\left(f_{\alpha_{\sigma(1)}}\right)^{\omega_1}\right)$$

$$\omega_2\alpha_{\sigma(2)}=\left(\sqrt{1-\left(1-t^2_{\alpha_{\sigma(2)}}\right)^{\omega_2}},1-\left(f_{\alpha_{\sigma(2)}}\right)^{\omega_2}\right).$$

Then

$$\omega_1\alpha_{\sigma(1)}\oplus\omega_2\alpha_{\sigma(2)}=\left(\sqrt{1-\prod_{j=1}^{2}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{2}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right).$$

Thus equation (6) holds for $n=2.$ Now we show that (6) holds for $n=k.$ i. e .,

$$\text{I-PFOWA}_\omega\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\dots,\langle u_k,\alpha_k\rangle\right)=\left(\sqrt{1-\prod_{j=1}^{k}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{k}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right). \tag{7}$$

Let us suppose that equation (6) holds for $n=k,$ then we show that (6) holds for $n=k+1.$

$$\text{I-PFOWA}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_{k+1},\alpha_{k+1}\rangle\right)$$
$$=\left(\sqrt{1-\prod_{j=1}^{k}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{k}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right)\oplus$$
$$\left(\sqrt{1-\left(1-t^2_{\alpha_{k+1}}\right)^{\omega_{k+1}}},1-\left(f_{\alpha_{k+1}}\right)^{\omega_{k+1}}\right)$$
$$=\left(\sqrt{1-\prod_{j=1}^{k+1}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{k+1}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right).$$

Hence equation (6) holds for $n=k+1$. Thus equation (6) holds for all n .

Example 1: Let

$$\langle u_1,\alpha_1\rangle=\langle 0.4,(0.6,0.7)\rangle,\langle u_2,\alpha_2\rangle=\langle 0.5,(0.7,0.6)\rangle$$
$$\langle u_3,\alpha_3\rangle=\langle 0.3,(0.5,0.8)\rangle,\langle u_4,\alpha_4\rangle=\langle 0.6,(0.8,0.5)\rangle$$
$$\langle u_5,\alpha_5\rangle=\langle 0.2,(0.4,0.9)\rangle$$

Performing the ordering with respect to the first element, then we have
$$\langle u_4,\alpha_4\rangle=\langle 0.6,(0.8,0.5)\rangle,\langle u_2,\alpha_2\rangle=\langle 0.5,(0.7,0.6)\rangle$$
$$\langle u_1,\alpha_1\rangle=\langle 0.4,(0.6,0.7)\rangle,\langle u_3,\alpha_3\rangle=\langle 0.3,(0.5,0.8)\rangle$$
$$\langle u_5,\alpha_5\rangle=\langle 0.2,(0.4,0.9)\rangle.$$

This ordering contains the ordered Pythagorean fuzzy arguments
$$\langle u_{\sigma(1)},\alpha_{\sigma(1)}\rangle=\langle 0.6,(0.8,0.5)\rangle,\langle u_{\sigma(2)},\alpha_{\sigma(2)}\rangle=\langle 0.5,(0.7,0.6)\rangle$$
$$\langle u_{\sigma(3)},\alpha_{\sigma(3)}\rangle=\langle 0.4,(0.6,0.7)\rangle,\langle u_{\sigma(4)},\alpha_{\sigma(4)}\rangle=\langle 0.3,(0.5,0.8)\rangle$$
$$\langle u_{\sigma(5)},\alpha_{\sigma(5)}\rangle=\langle 0.2,(0.4,0.9)\rangle$$

Let $\omega=(0.1,0.2,0.2,0.2,0.3)^T$ be the weight vector, then we have $t_{\alpha_{\sigma(1)}}=0.8,t_{\alpha_{\sigma(2)}}=0.7,t_{\alpha_{\sigma(3)}}=0.6,t_{\alpha_{\sigma(4)}}=0.5,t_{\alpha_{\sigma5}}=0.4,$ *and*

$$1-f_{\alpha_{\sigma(1)}}=0.5\Leftrightarrow f_{\alpha_{\sigma(1)}}=0.5$$
$$1-f_{\alpha_{\sigma(2)}}=0.6\Leftrightarrow f_{\alpha_{\sigma(2)}}=0.4$$
$$1-f_{\alpha_{\sigma(3)}}=0.7\Leftrightarrow f_{\alpha_{\sigma(3)}}=0.3$$
$$1-f_{\alpha_{\sigma(4)}}=0.8\Leftrightarrow f_{\alpha_{\sigma(4)}}=0.2$$
$$1-f_{\alpha_{\sigma(5)}}=0.9\Leftrightarrow f_{\alpha_{\sigma(5)}}=0.1$$

$$I-PFOWA_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\langle u_3,\alpha_3\rangle,\langle u_4,\alpha_4\rangle,\langle u_5,\alpha_5\rangle\right)$$
Thus
$$=\left[\sqrt{1-\prod_{j=1}^{5}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}},1-\prod_{j=1}^{5}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\right]$$
$$=\left[\begin{array}{l}\sqrt{1-(0.9028)(0.8740)(0.9146)(0.9440)(0.9440)},\\ 1-(0.9330)(0.8325)(0.7860)(0.7247)(0.5011)\end{array}\right]$$
$$=(0.597,0.778)$$

Theorem 2: (Commutativity): Let $\langle u_j,\alpha_j\rangle,\langle u_j,\alpha_j^*\rangle(j=1,2\ldots,n)$ be two set of 2- tuples, then

$$\begin{aligned}&\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\langle u_3,\alpha_3\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)\\&=\text{I-PFOWA}_{\omega}\left(\langle u_1^*,\alpha_1^*\rangle,\langle u_2^*,\alpha_2^*\rangle,\langle u_3^*,\alpha_3^*\rangle,\ldots,\langle u_n^*,\alpha_n^*\rangle\right)\end{aligned}\tag{8}$$

where $\left(\langle u_1^*,\alpha_1^*\rangle,\ldots,\langle u_n^*,\alpha_n^*\rangle\right)$ is any permutation of $\left(\langle u_1,\alpha_1\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)$.

Proof: As we know that

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\langle u_3,\alpha_3\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)$$
$$=\omega_1\alpha_{\sigma(1)}\oplus\omega_2\alpha_{\sigma(2)}\oplus\omega_3\alpha_{\sigma(3)}\oplus\ldots\oplus\omega_n\alpha_{\sigma(n)},$$

and

$$\text{I-PFOWA}_{\omega}\left(\langle u_1^*,\alpha_1^*\rangle,\langle u_2^*,\alpha_2^*\rangle,\langle u_3^*,\alpha_3^*\rangle,\ldots,\langle u_n^*,\alpha_n^*\rangle\right)$$
$$=\omega_1\alpha^*_{\sigma(1)}\oplus\omega_2\alpha^*_{\sigma(2)}\oplus\omega_3\alpha^*_{\sigma(3)}\oplus\ldots\oplus\omega_n\alpha^*_{\sigma(n)}.$$

Since $\left(\langle u_1^*,\alpha_1^*\rangle,\ldots,\langle u_n^*,\alpha_n^*\rangle\right)$ is any permutation of $\left(\langle u_1,\alpha_1\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)$, thus equation (8) always holds.

Theorem 3: (Idempotency): Let $\langle u_j,\alpha_j\rangle(j=1,2,3,\ldots,n)$ be a collection of 2-tuples, where $\alpha_j=\alpha$ for all j, then

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)=\alpha\tag{9}$$

Proof: As $\alpha_{\sigma(j)}=\alpha$ for all j, then we have

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)$$
$$=\left(\sqrt{1-\prod_{j=1}^{n}\left(1-t_{\alpha}^2\right)^{\omega_j}},1-\prod_{j=1}^{n}\left(f_{\alpha}\right)^{\omega_j}\right)$$
$$=\left(\sqrt{1-\left(1-t_{\alpha}^2\right)^{\sum_{j=1}^{n}\omega_j}},1-\left(f_{\alpha}\right)^{\sum_{j=1}^{n}\omega_j}\right)$$
$$=\left(\sqrt{1-\left(1-t_{\alpha}^2\right)},1-f_{\alpha}\right).$$

The proof is complete

Theorem 4: (Boundedness): Let $\langle u_j,\alpha_j\rangle(j=1,2,\ldots,n)$ be a collection of 2-tuples, and $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the weighted vector of α_j, with $\omega_j\in[0,1]$ and $\sum_{j=1}^{n}\omega_j=1$, then

$$\alpha_{\min}\le\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)\le\alpha_{\max},\tag{10}$$

Where

$$\alpha_{\min}=\left(\min_j\left(t_{\alpha_{\sigma(j)}}\right),1-\max_j\left(f_{\alpha_{\sigma(j)}}\right)\right) \quad (11)$$

$$\alpha_{\max}=\left(\max_j\left(t_{\alpha_{\sigma(j)}}\right),1-\min_j\left(f_{\alpha_{\sigma(j)}}\right)\right) \quad (12)$$

Proof: Since

$$\min_j\left(t_{\alpha_{\sigma(j)}}\right)\le t_{\alpha_{\sigma(j)}}\le\max_j\left(t_{\alpha_{\sigma(j)}}\right) \quad (13)$$

$$\min_j\left(f_{\alpha_{\sigma(j)}}\right)\le f_{\alpha_{\sigma(j)}}\le\max_j\left(f_{\alpha_{\sigma(j)}}\right) \quad (14)$$

From equation (13), we have

$$\begin{aligned}&\Leftrightarrow\sqrt{\prod_{j=1}^{n}\left(1-\max_j\left(t_{\alpha_{\sigma(j)}}\right)^2\right)^{\omega_j}}\le\sqrt{\prod_{j=1}^{n}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}}\le\sqrt{\prod_{j=1}^{n}\left(1-\min_j\left(t_{\alpha_{\sigma(j)}}\right)^2\right)^{\omega_j}}\\&\Leftrightarrow\sqrt{\left(1-\max_j\left(t_{\alpha_{\sigma(j)}}\right)^2\right)^{\sum_{j=1}^{n}\omega_j}}\le\sqrt{\prod_{j=1}^{n}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}}\le\sqrt{\left(1-\min_j\left(t_{\alpha_{\sigma(j)}}\right)^2\right)^{\sum_{j=1}^{n}\omega_j}}\\&\Leftrightarrow\min_j\left(t_{\alpha_{\sigma(j)}}\right)\le\sqrt{1-\prod_{j=1}^{n}\left(1-t^2_{\alpha_{\sigma(j)}}\right)^{\omega_j}}\le\max_j\left(t_{\alpha_{\sigma(j)}}\right)\end{aligned} \quad (15)$$

Now from equation (14), we have

$$\begin{aligned}&\Leftrightarrow\prod_{j=1}^{n}\min_j\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\le\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\le\prod_{j=1}^{n}\max_j\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\\&\Leftrightarrow\min_j\left(f_{\alpha_{\sigma(j)}}\right)^{\sum_{j=1}^{n}\omega_j}\le\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\le\max_j\left(f_{\alpha_{\sigma(j)}}\right)^{\sum_{j=1}^{n}\omega_j}\\&\Leftrightarrow\min_j\left(f_{\alpha_{\sigma(j)}}\right)\le\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\le\max_j\left(f_{\alpha_{\sigma(j)}}\right).\\&\Leftrightarrow 1-\max_j\left(f_{\alpha_{\sigma(j)}}\right)\le1-\prod_{j=1}^{n}\left(f_{\alpha_{\sigma(j)}}\right)^{\omega_j}\le1-\min_j\left(f_{\alpha_{\sigma(j)}}\right)\end{aligned} \quad (16)$$

Let

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)=\alpha=\left(t_{\alpha_j},1-f_{\alpha_j}\right) \quad (17)$$

then

$$S(\alpha)=t^2_{\alpha_j}-f^2_{\alpha_j}\le\max_j\left(t_{\alpha_j}\right)^2-\min_j f^2=S(\alpha_{\max}) \quad (18)$$

Again

$$S(\alpha)=t^2_{\alpha_j}-f^2_{\alpha_j}\ge\min_j\left(t_{\alpha_j}\right)^2-\max_j\left(f_{\alpha_j}\right)^2=S(\alpha_{\min}) \quad (19)$$

if

$$S(\alpha)<S(\alpha_{\max}),S(\alpha)>S(\alpha_{\min}),$$

then

$$\alpha_{\min}<\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)<\alpha_{\max} \quad (20)$$

If

$$\begin{aligned}&\Leftrightarrow S(\alpha)=S(\alpha_{\max})\\&\Leftrightarrow\lambda^2_{\alpha_j}-\eta^2_{\alpha_j}=\max_j\left(\lambda_{\alpha_j}\right)^2-\min_j\left(\eta_{\alpha_j}\right)^2\\&\Leftrightarrow\lambda_{\alpha_j}=\max_j\left(\lambda_{\alpha_j}\right),\eta_{\alpha_j}=\min_j\left(\eta_{\alpha_j}\right)\\&\Leftrightarrow\lambda^2_{\alpha_j}+\eta^2_{\alpha_j}=\max_j\left(\lambda_{\alpha_j}\right)^2+\min_j\left(\eta_{\alpha_j}\right)^2\\&\Leftrightarrow H(\alpha)=H(\alpha_{\max}).\end{aligned} \quad (21)$$

Thus from equation (21) we have

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)=\alpha_{\max}. \quad (22)$$

If

$$\begin{aligned}&\Leftrightarrow S(\alpha)=S(\alpha_{\min})\\&\Leftrightarrow t^2_{\alpha_j}-f^2_{\alpha_j}=\min_j\left(f_{\alpha_j}\right)^2-\max_j\left(t_{\alpha_j}\right)^2\\&\Leftrightarrow t_{\alpha_j}=\min_j\left(f_{\alpha_j}\right),f_{\alpha_j}=\max_j\left(t_{\alpha_j}\right)\\&\Leftrightarrow t^2_{\alpha_j}+f^2_{\alpha_j}=\min_j\left(f_{\alpha_j}\right)^2+\max_j\left(t_{\alpha_j}\right)^2\\&\Leftrightarrow H(\alpha)=H(\alpha_{\min}).\end{aligned} \quad (23)$$

Thus from equation (23) we have

$$\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)=\alpha_{\min} \quad (24)$$

Thus from equation (20) to (24), we have equation (10) always holds

Theorem 5: (Monotonicity): Let $\langle u_j,\alpha_j\rangle,\langle u_j,\alpha^*_j\rangle(j=1,\ldots,n)$ be two set of 2-tuples, where $\alpha_{\sigma(j)}<\alpha^*_{\sigma(j)}$ for all j, then

$$\begin{aligned}&\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)\\&\le\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha^*_1\rangle,\langle u_2,\alpha^*_2\rangle,\ldots,\langle u_n,\alpha^*_n\rangle\right)\end{aligned} \quad (25)$$

Proof: As we know that

$$\begin{aligned}&\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,\ldots,\langle u_n,\alpha_n\rangle\right)\\&=\omega_1\alpha_{\sigma(1)}\oplus\omega_2\alpha_{\sigma(2)}\oplus\ldots\oplus\omega_n\alpha_{\sigma(n)},\end{aligned}$$

and

$$\begin{aligned}&\text{I-PFOWA}_{\omega}\left(\langle u_1,\alpha^*_1\rangle,\langle u_2,\alpha^*_2\rangle\ldots,\langle u_n,\alpha^*_n\rangle\right)\\&=\omega_1\alpha^*_{\sigma(1)}\oplus\omega_2\alpha^*_{\sigma(2)}\oplus\ldots\oplus\omega_n\alpha^*_{\sigma(n)}.\end{aligned}$$

Since $\alpha_{\sigma(j)}\le\alpha^*_{\sigma(j)}$ for all j, thus equation (25) always holds.

3.2. Induced Pythagorean Fuzzy Hybrid Averaging Operator

Definition 4: An induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator can be define as:

$$\text{I-PFHA}_{\omega,\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,...,\langle u_n,\alpha_n\rangle\right) = \left(\sqrt{1-\prod_{j=1}^{n}\left(1-t^2_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}}, 1-\prod_{j=1}^{n}\left(f_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}\right) \quad (26)$$

where $\dot{\alpha}_{\sigma(j)}$ is the weighted Pythagorean fuzzy value $\dot{\alpha}_{\sigma(j)}\left(\dot{\alpha}_{\sigma(j)} = n\omega_j\alpha_{\sigma(j)}, j=1,...,n\right)$ of the Pythagorean fuzzy ordered weighted averaging pair $\langle u_j,\alpha_j\rangle$ having the j^{th} largest u_j, u_j in $\langle u_j,\alpha_j\rangle$ is the order inducing variable and α_j is the Pythagorean fuzzy argument variable $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weighted vector of $\alpha_j\ (j=1,...,n)$ and also $\omega_j \in [0,1], \sum_{j=1}^{n}\omega_j = 1.$

Theorem 6: Let $\langle u_j,\alpha_j\rangle (j=1,2,...,n)$ be a collection of 2-tuples, then their aggregated value by using the I-PFHA operator is also a Pythagorean fuzzy value, and

$$\text{I-PFHA}_{\omega,\omega}\left(\langle u_1,\alpha_1\rangle,\langle u_2,\alpha_2\rangle,...,\langle u_n,\alpha_n\rangle\right) = \left(\sqrt{1-\prod_{j=1}^{n}\left(1-t^2_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}}, 1-\prod_{j=1}^{n}\left(f_{\dot{\alpha}_{\sigma(j)}}\right)^{\omega_j}\right) \quad (27)$$

where $\omega=(\omega_1,\omega_2,...,\omega_n)^T$ is the weighted vector of $\alpha_j\ (j=1,2,3,...,n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1.$

Proof: Proof is similar to Theorem 1.

4. Conclusion

In this paper, we have familiarized the idea of induced Pythagorean fuzzy ordered weighted averaging (I-PFOWA) operator and induced Pythagorean fuzzy hybrid averaging (I-PFHA) operator and also discussed some of their basic properties.

References

[1] Atanassov K. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* 1986, K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst, (1986), 87-96.

[2] L. A. Zadeh, Fuzzy sets, Inf Control, (1965), 338-353.

[3] H. Bustine and P. Burillo, Vague sets are intuitionistic fuzzy sets. Fuzzy sets and systems, (1996), 79 (3), 403–405.

[4] C. H. Tan and X. H. Chen, Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making, Expert Syst Appl, (2010), 149.157.

[5] D. H. Hong, and C. H. Choi, Multicriteria fuzzy decision-making problems based on vague set theory. Fuzzy sets and systems, (2000) 114 (1), 103–113.

[6] H. Bustine and P. Burillo, Vague sets are intuitionistic fuzzy sets. Fuzzy sets and systems, (1996) 79 (3), 403–405.

[7] K. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets Syst, (1994), 137-142.

[8] K. Atanassov, Remarks on the intuitionistic fuzzy sets. III, Fuzzy Sets Syst, (1995), 401-402.

[9] K. Atanassov, equality between intuitionistic fuzzy sets, Fuzzy Sets Syst, (1996), 257-258.

[10] K. Atanassov, Intuitionistic fuzzy sets: theory and applications, Heidelberg, Germany: Physica-Verlag (1999).

[11] M. Xia and Z. S. Xu, Generalized point operators for aggregating intuitionistic fuzzy information, Int J Intell Syst (2010), 1061-1080.

[12] S. K. De, R. Biswas and A. R. Roy, Some operations on intuitionistic fuzzy sets, Fuzzy Set Syst, (2000), 477-484.

[13] Z. S. Xu, Intuitionistic fuzzy aggregation operators. IEEE Trans Fuzzy Syst, (2007), 1179- 1187.

[14] Z. S. Xu, R. R. Yager. Some geometric aggregation operators based on intuitionistic fuzzy sets, Int J Gen Syst (2006), 417-433.

[15] W. Wang and X. Liu, Intuitionistic Fuzzy Geometric Aggregation Operators Based on Einstein Operations, international journal of intelligent systems, (2011), 1049-1075.

[16] Weize Wang, Xinwang Liu, Intuitionistic Fuzzy Information Aggregation Using Einstein Operations, IEEE Trans. Fuzzy Systems, (2012) 923-938.

[17] X. Zhao and G. Wei, Some intuitionistic fuzzy Einstein hybrid aggregation operators And their application to multiple attribute decision making, Knowledge-Based Systems, (2013). 472-479.

[18] R. R. Yager, Pythagorean fuzzy subsets, In Proc. Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada (2013), 57-61.

[19] R. R. Yager, A. M. Abbasov, Pythagorean membership grades, complex numbers and decision making. Int J Intell Syst (2013), 28:436.452.

[20] K. Rahman, S. Abdullah, M. S. Ali Khan, A. Ali and F. Amin, Pythagorean fuzzy hybrid averaging aggregation operator and its application to multiple attribute decision making. Accepted.

[21] K. Rahman, M. S. Ali. Khan, Murad Ullah and A. Fahmi, Multiple attribute group decision making for plant location selection with Pythagorean fuzzy weighted geometric aggregation operator, The Nucleus (2017), 54, 66-74.

[22] K. Rahman, S. Abdullah, F. Husain M. S. Ali Khan, M. Shakeel, Pythagorean fuzzy ordered weighted geometric aggregation operator and their application to multiple attribute group decision making, J. Appl. Environ. Biol. Sci., (2017), 7(4) 67-83.

[23] K. Rahman, S. Abdullah, M. S. Ali Khan and M. Shakeel, Pythagorean fuzzy hybrid geometric aggregation operator and their applications to multiple attribute decision making, International Journal of Computer Science and Information Security (IJCSIS), (2016), 14, No. 6, 837-854.

[24] H. Garg, A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making, international journal of intelligent systems, (2016), 1-35.