

Some Common Fixed Point Results in Cone Metric Spaces for Rational Contractions

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Abstract: A very engrossing technique in theory of contractive mapping fixed point. A number of authors have defined contractive type mappings on a cone metric spaces X which are generalization of the well-known Banach contraction, and have the property that each of such mapping has a unique fixed point. The fixed point can always be found by using Picard Iteration, opening with initial choice $x_0 \in X$. In this manuscript, we generalize, extend and improve the result under the assumption of normality of cone for rational expression type contraction mapping in cone metric spaces. The present article is to provide a new alternative proof for two and three mapping and obtain the entity and exclusiveness of common fixed point. The concernment of the present paper to open a new direction of proof to be extended based on the methods of rational type contraction mapping in cone metric spaces of fixed-point theory. The assistance of this article is organized as follows. In section 2, preliminary notes. In this section we recall some standard notations and definitions which we needed. In section 3, the main results of the author are given. In this section we evidence of new results for two and three maps. In section 4, gives brief concluding note of the paper.

Keywords: Fixed Point, Common Fixed Point, Cone Metric Space, Rational Expression

1. Introduction

The first fundamental result on fixed point for contractive mapping was published and introduced by Banach, S. in 1922, which is known as Banach contraction principle, which uses the concept of Lipschitz mappings [1]. The famous definition and theorem run as follows:

Definition 1.1. Let (X, d) be a metric space. The mapping $T: X \rightarrow X$ is said to be Lipschitzian, if there exist a constant $k > 0$ (called Lipschitz constant) such that

$d(Tx, Ty) \leq kd(x, y)$, for all $x, y \in X$. A Lipschitzian mapping with a Lipschitz constant $k < 1$ is called contraction.

Theorem 1.2. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a contraction mapping with Lipschitz constant $k < 1$. Then T has a unique fixed point w in X , and for each $x \in X$ we have:

$$\lim_{n \rightarrow \infty} T^n(x_0) = w.$$

Moreover, for each $x \in X$, we have:

$$d(T^n x, w) \leq \frac{k^n}{1-k} d(Tx, x).$$

This theorem provides an impressive illustration of the unifying power of functional analytic methods and their usefulness in various disciplinary. It has become a very popular source of existence and uniqueness theorems in different branches of Mathematical analysis. Banach contraction principle was the only main tools, which is used by Kannan in 1968 & 1969 to establish the existence and uniqueness of fixed point and introduced the concept of Kannan contractive mappings [22, 23] as follows:

Theorem 1.3. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be Kannan contraction mapping, if there exist a constant $b \in [0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$. Then T has a unique fixed point.

Chatterjea in 1972 introduced the concept of chatterjea contraction mapping [24], as follows:

Theorem 1.4. Let $T: X \rightarrow X$ be Chatterjea contraction mapping on complete metric space (X, d) and if there exist a constant $c \in [0, \frac{1}{2})$ such that:

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$$

for all $x, y \in X$. Then T has a unique fixed point.

In 1972, Zamfirescu Introduced the concept of Zamfirescu mapping as follows:

Theorem 1.5. Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a Zamfirescu contraction mapping, if there exist a constant $\alpha \in [0, 1), \beta \in [0, \frac{1}{2})$ and $\gamma \in [0, \frac{1}{2})$ such that at least one of the following conditions is true.

$$(z_1) d(Tx, Ty) \leq \alpha d(x, y),$$

$$(z_2) d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)],$$

$$(z_3) d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)],$$

for all $x, y \in X$. Then T has a unique fixed point.

After that, a number of authors have defined contractive type mappings on a complete metric space X . The first result on fixed points for contractive type mapping was the much-celebrated Banach contraction principle with rational expressions have been expanded and some fixed-point results have been obtained in [3-6].

Recently, Huang and Zhang [7] generalized the concept of metric space by replacing the set of real numbers by ordered Banach space and obtained some fixed-point theorems of contractive mappings in cone metric spaces. Subsequently, several other authors see for instance [8-18, 26-32] generalized and studied fixed and common fixed-point results in cone metric spaces for normal and non-normal.

Quite recently, R. Uthaya Kumar et al. [19] proved common fixed-point theorem in cone metric space for rational contraction, which generalization of the main result of Arshad et al. [20]. The result of [19] is also generalized by Tiwari, S. K. et al. [21]. In 2017, Pawan Kumar and Ansari [2], proved and generalized some fixed-point result in cone metric spaces for rational expression in normal cone setting, which is the generalize the main result of Das & Gupta [3].

So, the aim of this paper is to prove common fixed-point theorem in cone metric spaces for rational expression by normality of cone. Our results improve, extend and generalize the main result of Kumar, P. and Ansari, Z. K. [2].

The assistance of this article is organized as follows. In section 2, preliminary notes. In this section we recall some standard notations and definitions which we needed. In section 3, the main results of the author are given. In this section we evidence of new results of the result of Kumar, P. and Ansari, Z. K. [2] for two and three maps. In section 4, gives brief concluding note of the paper.

2. Preliminaries Notes

Definition 2.1 [7]: Let E be a real Banach space and P be a subset of E and 0 denote to the zero element in E , then P is called a cone if and only if:

P is a non-empty set closed and $P \neq \{0\}$,

If a, b are non-negative real numbers and $x, y \in P$, then $ax + by \in P$,

$x \in P$ and $-x \in P \Rightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering \leq on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in \text{int}P$ (where $\text{int}P$ denotes the interior of P). If $\text{int}P \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \|x\| \leq M \|y\|.$$

The least positive number k satisfying the above is called the normal constant of P .

Definition 2.2 [7]: Let X be a non-empty set. Suppose E is a real Banach space, P is a cone with $\text{int}P \neq \emptyset$ and \leq is a partial ordering with respect to P . If the mapping $d: X \times X \rightarrow E$ satisfies.

$0 < d(x, y)$ for all $x, y \in X$ and $(x, y) = 0$ if and only if $x = y$,

$$d(x, y) = d(y, x) \text{ for all } x, y \in X,$$

$$d(x, y) \leq d(x, z) + d(z, y) \text{ for all } x, y, z \in X.$$

Then d is called a cone metric on X , and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of a metric space.

Example 2.3: Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E: x, y \geq 0\}$, $X = \mathbb{R}$ and $d: X \times X \rightarrow E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition: 2.4 [7]: Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n \geq 1}$ be a sequence in X . Then,

1. $\{x_n\}_{n \geq 1}$ Converges to x whenever for every $c \in E$ with $\theta \ll c$, if there is a natural Number N such That $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x, (n \rightarrow \infty)$.

2. $\{x_n\}_{n \geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $\theta \ll c$, if there is Natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

3. (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Lemma 1.10: Let (X, d) be a cone metric space and $\{x_{2n}\} \in X$ such that.

$$d(x_{2n+1}, x_{2n}) \leq kd(x_{2n}, x_{2n-1}),$$

where $0 \leq k < 1$.

Then $\{x_{2n}\}$ be a Cauchy sequence.

3. Main Result

In this section, we obtain and generalize common fixed-point theorems for two and three self-mappings rational expression in cone metric spaces by assumption normality of

cone, which is given by Kumar. and Ansari, Z. K. [2]. The Kumar, P. and Ansari, Z. K (2017). proved the following theorem:

Theorem 3.1: Let (X, d) be a complete cone metric space. P a normal cone with normal constant M .

A self-mapping $T: X \times X \rightarrow X$ satisfies the condition:

$$\begin{aligned} d(Tx, Ty) &\leq \alpha_1 d(x, y) \\ &+ \alpha_2 [d(x, Tx) + d(y, Ty)] \\ &\frac{[d(x, y) + d(y, Ty)]}{[d(x, Ty)]} \\ &+ \alpha_3 [d(x, Ty) + d(y, Tx)] \\ &\frac{[d(x, y) + d(y, Ty) + d(x, Ty)]^2}{[d(x, Ty)]^2}, \end{aligned}$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 > 0$ With $\alpha_1 + 2\alpha_2 + 8\alpha_3 \leq 1$. Then T has a unique fixed point in X .

Theorem 3.2: Let (X, d) be a complete cone metric space. P a normal cone with normal constant M . A self-mapping $T: X \times X \rightarrow X$ satisfies the condition:

$$\begin{aligned} d(Tx, Ty) &\leq \alpha_1 d(x, y) \\ &+ \alpha_2 \frac{[d(x, Ty)]^2}{[d(x, y) + d(y, Ty)]} + \alpha_3 \frac{[d(y, Ty)]^2}{[d(x, y) + d(x, Ty)]} \end{aligned}$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 > 0$ With $\alpha_1 + 2\alpha_2 + 8\alpha_3 \leq 1$. then T has a unique fixed point in X .

Now, first we will derive and generalize the Theorem 3.1 & Theorem 3.2 for two self-mapping in cone metric space as follows:

Theorem 3.3: Let (X, d) be a complete cone metric space. P a normal cone with normal constant K . Two self-mapping $P, Q: X \times X \rightarrow X$ satisfies the condition:

$$\begin{aligned} d(P\rho, Q\sigma) &\leq ad(\rho, \sigma) \\ &+ b[d(\rho, P\rho) + d(\sigma, Q\sigma)] \cdot \frac{[d(\rho, \sigma) + d(\sigma, Q\sigma)]}{[d(\rho, Q\sigma)]} \\ &+ c[d(\rho, Q\sigma) + d(\sigma, P\rho)]. \\ &\frac{[d(\rho, \sigma) + d(\sigma, Q\sigma) + d(\rho, Q\sigma)]^2}{[d(\rho, Q\sigma)]^2} \dots \end{aligned} \quad (1)$$

for all $\rho, \sigma \in X$ where $a, b, c > 0$ With $a + 2b + 8c \leq 1$. then P and Q have a unique common fixed point in X .

Proof: Let ρ_0 be an arbitrary point in X . We define the iterative sequence $\{\rho_{2n}\}$ and $\{\rho_{2n+1}\}$ by:

$$\rho_{2n+1} = P\rho_{2n} = P^{2n}\rho_0$$

And:

$$\rho_{2n+2} = Q\rho_{2n+1} = Q^{2n+1}\rho_0, \text{ for each}$$

$$n = 0, 1, 2, 4 \dots \dots \dots \infty.$$

Put $\rho = \rho_{2n-1}$ and $\sigma = \rho_{2n}$ in (1) we have

$$\begin{aligned} d(\rho_{2n}, \rho_{2n+1}) &= d(P\rho_{2n-1}, Q\rho_{2n}) \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \end{aligned}$$

$$\begin{aligned} &+ b[\rho_{2n-1}, P\rho_{2n-1}) + d(\rho_{2n}, Q\rho_{2n})] \\ &\times \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, Q\rho_{2n})]}{[d(\rho_{2n-1}, Q\rho_{2n})]} \\ &+ c[d(\rho_{2n-1}, Q\rho_{2n}) + d(Q\rho_{2n}, P\rho_{2n-1})] \\ &\times \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, Q\rho_{2n}) + d(\rho_{2n-1}, Q\rho_{2n})]^2}{[d(\rho_{2n-1}, Q\rho_{2n})]^2} \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \\ &+ b[\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})] \\ &\times \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]}{[d(\rho_{2n-1}, \rho_{2n+1})]} \\ &+ c[d(\rho_{2n-1}, \rho_{2n+1}) + d(\rho_{2n+1}, \rho_{2n})] \\ &\times \frac{[d(\rho_{2n-1}, \sigma_{2n}) + d(\rho_{2n}, \rho_{2n+1}) + d(\rho_{2n-1}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n+1})]^2} \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \\ &+ b[\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})] \\ &\times \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]} \\ &+ c[d(\rho_{2n-1}, \rho_{2n+1}) + d(\rho_{2n}, \rho_{2n+1})] \\ &\times \frac{[d(\rho_{2n-1}, \sigma_{2n}) + d(\rho_{2n}, \rho_{2n+1}) + d(\rho_{2n-1}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]^2} \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) + b[\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})] \\ &+ 4c[d(\rho_{2n-1}, \rho_{2n+1}) + d(\rho_{2n}, \rho_{2n+1})]. \\ &\text{Therefore, } (1 - b - 4c)d(\rho_{2n}, \rho_{2n+1}) \\ &\leq (a + b + 4c)d(\rho_{2n-1}, \rho_{2n}) \\ &\Rightarrow d(\rho_{2n}, \rho_{2n+1}) \leq \frac{(a+b+4c)}{(1-b-4c)} d(\rho_{2n-1}, \rho_{2n}) \\ &\leq rd(\rho_{2n-1}, \rho_{2n}), \end{aligned}$$

where $r = \frac{(a+b+4c)}{(1-b-4c)} < 1, 0 < r < 1$.

In general, by induction we have

$$\begin{aligned} d(\rho_{2n}, \rho_{2n+1}) &\leq rd(\rho_{2n-1}, \rho_{2n}) \\ &\leq \dots \dots \dots \leq \\ &\leq r^{2n} d(\rho_0, \rho_1). \end{aligned}$$

So, for $m, n \in N$ with $n > m$ we have

$$\begin{aligned} d(\rho_{2n}, \rho_{2m}) &\leq d(\rho_{2n}, \rho_{2n-1}) \\ &+ d(\rho_{2n-1}, \rho_{2n-2}) + \dots \dots \dots + d(\rho_{2n+2m+1}, \rho_{2m}) \\ &\leq (r^{2n+1} + r^{2n+2} + \dots) d(\rho_0, \rho_1), \\ &\leq \frac{r^{2n}}{1-r} d(\rho_0, \rho_1), \dots \end{aligned} \quad (2)$$

By normality of cone, so by (2) we get

$\|d(\rho_{2n}, \rho_{2m})\| \leq M \frac{r^{2n}}{1-r} \|d(\rho_0, \rho_1)\|$, which implies that $d(\rho_{2n}, \rho_{2m}) \rightarrow 0$ as $m, n \rightarrow \infty$. Thus $\{\rho_{2n}\}$ is a Cauchy sequence in X . Since (X, d) is a complete cone metric spaces, there exist $u \in X$ Such

$$\rho_{2n} \rightarrow u \ (n \rightarrow \infty).$$

On the other hand

$$\begin{aligned} d(u, Pu) &\leq [d(u, \rho_{2n+1}) + d(\rho_{2n+1}, Pu)] \\ &\leq [d(u, \rho_{2n+1}) + d(\rho_{2n}, Pu)] \\ &\leq [d(u, P\rho_{2n}) + d(P\rho_{2n}, Pu)] \\ &\leq d(u, P\rho_{2n}) + ad(\rho_{2n}, u) \\ &+ b[(\rho_{2n} P\rho_{2n}) + d(u, Pu)] \times \frac{[d(\rho_{2n}, u) + d(u, Pu)]}{[d(\rho_{2n}, u) + d(u, Pu)]} \\ &+ c[d(\rho_{2n}, Pu) + d(u, Pu)] \\ &\times \frac{[d(\rho_{2n}, u) + d(u, Pu) + d(\rho_{2n}, u) + d(u, Pu)]^2}{[d(\rho_{2n}, u) + d(u, Pu)]^2}. \end{aligned}$$

Therefore

$$\begin{aligned} d(u, Pu) &\leq d(u, \rho_{2n+1}) + ad(\rho_{2n}, u) \\ &+ b[d(\rho_{2n}, \rho_{2n+1}) + d(u, Pu)] \\ &+ 4c[d(\rho_{2n}, Pu) + d(u, Pu)] \dots (3) \end{aligned}$$

by the condition of normality of cone in (3), we get

$$\begin{aligned} \|d(u, Pu)\| &\leq M(\|d(u, \rho_{2n+1})\| + a\|d(\rho_{2n}, u)\|) \\ &+ b\|[d(\rho_{2n}, \rho_{2n+1}) + d(u, Pu)]\| \\ &+ 4c\|[d(\rho_{2n}, Pu) + d(u, Pu)]\| \rightarrow 0 \\ &\leq M(\|d(u, \rho_{2n+1})\| + a\|d(\rho_{2n}, u)\|) \\ &+ b\|[d(\rho_{2n}, \rho_{2n+1})\| + \|d(u, Pu)\|] \\ &+ 4c\|[d(\rho_{2n}, Pu)\| + \|d(u, Pu)\|] \rightarrow 0. \end{aligned}$$

as $n \rightarrow \infty$. Hence $\|d(u, Pu)\| \leq 0$. Thus $d(u, Pu) = 0$.

This implies $u = Pu$. So u is fixed point of P .

Similarly, it can be established that, u is also fixed point of Q , that is, $u = Qu$. Hence $Pu = u = Qu$. Therefore, u is common fixed point of P and Q .

Now to prove uniqueness: Suppose that v is another common fixed point of P and Q such that then $Pv = v = Qv$. Now from (1) we have

$$\begin{aligned} d(u, v) &= d(Pu, Qv) \\ &\leq a(u, v) + b[d(u, Pu) + d(v, Qv)] \\ &\times \frac{[d(u, v) + d(v, Qv)]}{[d(u, Qv)]} + c[d(u, Qv) + d(v, Pu)] \end{aligned}$$

$$\begin{aligned} &\times \frac{[d(u, v) + d(v, Qv) + d(u, Qv)]^2}{[d(u, Qv)]^2} \\ &\leq a(u, v) + b[d(u, u) + d(v, v)] \\ &\times \frac{[d(u, v) + d(v, v)]}{[d(u, v)]} + c[d(u, v) + d(v, u)] \\ &\times \frac{[d(u, v) + d(v, v) + d(u, v)]^2}{[d(u, v)]^2} \\ &\leq a(u, v) + c[2d(u, v)] \times \frac{[2d(u, v)]^2}{[d(u, v)]^2} \\ &\leq (a + 8c)d(u, v) \end{aligned}$$

Hence

$\|d(u, v)\| \leq M[(a + 8c)\|d(u, v)\|] \rightarrow 0$ as $n \rightarrow \infty$. So, we have $\|d(u, v)\| = 0$. Implies that $u = v$. So, $u = v$ is a unique common fixed point of P and Q . This completes the proof of theorem.

Next, we prove and generalize the theorem 3.2. for two self-mapping.

Theorem 3.4: Let (X, d) be a complete cone metric space. P a normal cone with normal constant M . The two self-mapping $P, Q: X \times X \rightarrow X$ satisfies the condition

$$\begin{aligned} d(P\rho, Q\sigma) &\leq ad(\rho, \sigma) + b \frac{[d(\rho, Qv)]^2}{[d(\rho, \sigma) + d(\sigma, Q\sigma)]} \\ &+ c \frac{[d(\sigma, Q\sigma)]^2}{[d(\rho, \sigma) + d(\rho, Q\sigma)]} \dots \end{aligned} \quad (4)$$

for all $\rho, \sigma \in X$ where $a, b, c > 0$ With $a + 2b + 4c \leq 1$. then, P and Q have a unique common fixed point in X .

Proof: Let ρ_0 be an arbitrary point in X . We define the iterative sequence $\{\rho_{2n}\}$ and $\{\rho_{2n+1}\}$ by

$$\rho_{2n+1} = P\rho_{2n} = P^{2n}\rho_0$$

And

$$\rho_{2n+2} = Q\rho_{2n+1} = Q^{2n+1}\rho_0,$$

for each $n = 0, 1, 2, 4 \dots \dots \dots \infty$.

Put $\rho = \rho_{2n-1}$ and $\sigma = \rho_{2n}$ in (4) we have

$$\begin{aligned} d(\rho_{2n}, \rho_{2n+1}) &= d(P\rho_{2n-1}, Q\rho_{2n}) \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \\ &+ b \frac{[d(\rho_{2n-1}, Q\rho_{2n})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, Q\rho_{2n})]} \\ &+ c \frac{[d(\rho_{2n}, Q\rho_{2n})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n-1}, Q\rho_{2n})]} \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \\ &+ b \frac{[d(\rho_{2n-1}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2+1})]} \\ &+ c \frac{[d(\rho_{2n}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n-1}, \rho_{2n+1})]} \\ &\leq ad(\rho_{2n-1}, \rho_{2n}) \end{aligned}$$

$$\begin{aligned}
& + b \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})]} \\
& + c \frac{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n-1}, \rho_{2n+1})]^2}{[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n-1}, \rho_{2n+1})]} \\
& \leq ad(\rho_{2n-1}, \rho_{2n}) \\
& + b[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})] \\
& + c[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n-1}, \rho_{2n+1})] \\
& \leq ad(\rho_{2n-1}, \rho_{2n}) \\
& + b[d(\rho_{2n-1}, \rho_{2n}) + d(\rho_{2n}, \rho_{2n+1})] \\
& + c[d(\rho_{2n-1}, \rho_{2n}) + \{d(\rho_{2n-1}, \rho_{2n}) \\
& + d(\rho_{2n}, \rho_{2n+1})\}]
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (1 - b - c)d(\rho_{2n}, \rho_{2n+1}) \\
& \leq (a + b + c)d(\rho_{2n-1}, \rho_{2n})
\end{aligned}$$

Hence

$$d(\rho_{2n}, \rho_{2n+1}) \leq \frac{(a+b+2c)}{(1-b-c)} d(\rho_{2n-1}, \rho_{2n})$$

Implies that,

$$\begin{aligned}
& d(\rho_{2n}, \rho_{2n+1}) \leq \xi d(\rho_{2n-1}, \rho_{2n}), \text{ where} \\
& \xi = \frac{(a+b+2c)}{(1-b-c)} < 1, 0 < \xi < 1.
\end{aligned}$$

In general, by induction we have

$$\begin{aligned}
& d(\rho_{2n}, \rho_{2n-1}) \leq \xi d(\rho_{2n-1}, \rho_{2n}) \\
& \leq \dots \leq \\
& \leq \xi^{2n} d(\rho_0, \rho_1)
\end{aligned}$$

So, for $m, n \in N$ with $n > m$, we have

$$\begin{aligned}
& d(\rho_{2n}, \rho_{2m}) \leq d(\rho_{2n}, \rho_{2n-1}) + d(\rho_{2n-1}, \rho_{2n-2}) \\
& + \dots + d(\rho_{2n+2m+1}, \rho_{2m}) \\
& \leq (\xi^{2n+1} + \xi^{2n+2} + \dots + \xi^{2n+2m-1})d(\rho_0, \rho_1), \\
& \leq \frac{\xi^{2n}}{1-\xi} d(\rho_0, \rho_1) \dots \dots \dots (5)
\end{aligned}$$

Since P is normal cone with normal constant, so by (5) we get

$$\|d(\rho_{2n}, \rho_{2m})\| \leq M \frac{\xi^{2n}}{1-\xi} \|d(\rho_0, \rho_1)\| \rightarrow 0.$$

Implies that, $d(\rho_{2n}, \rho_{2m}) \rightarrow 0$ as $m, n \rightarrow \infty$. Thus $\{\rho_{2n}\}$ is a Cauchy sequence in X . Since (X, d) is a complete cone metric spaces, there exist $u^* \in X$ Such that:

$$\rho_{2n} \rightarrow u^* (n \rightarrow \infty).$$

On the other hand

$$\begin{aligned}
& d(u^*, Pu^*) \leq [d(u^*, \rho_{2n+1}) + d(\rho_{2n+1}, Pu^*)] \\
& \leq [d(u^*, \rho_{2n+1}) + d(P\rho_{2n}, Pu^*)] \\
& \leq [d(u^*, P\rho_{2n}) + d(P\rho_{2n}, Pu^*)] \\
& \leq d(u^*, P\rho_{2n}) + ad(u^*, \rho_{2n}) \\
& + b \frac{[d(\rho_{2n}, Pu^*)]^2}{[d(\rho_{2n}, u^*) + d(u^*, Pu^*)]} \\
& + c \frac{[d(u^*, Pu^*)]^2}{[d(\rho_{2n}, u^*) + d(u^*, Pu^*)]} \\
& \leq d(u^*, \rho_{2n+1}) + ad(u^*, \rho_{2n}) \\
& + b \frac{[d(\rho_{2n}, u^*) + d(u^*, Pu^*)]^2}{[d(\rho_{2n}, u^*) + d(u^*, Pu^*)]} \\
& + c \frac{[d(u^*, Pu^*)]^2}{[d(u^*, Pu^*)]} \\
& \leq d(u^*, \rho_{2n+1}) + ad(u^*, \rho_{2n}) \\
& + b[d(\rho_{2n}, u^*) + d(u^*, Pu^*)] \\
& + c d(u^*, Pu^*)
\end{aligned}$$

Now using the condition of normality of cone

$$\begin{aligned}
& \|d(u^*, Pu^*)\| \leq M(\|d(u^*, \rho_{2n+1})\| + a\|d(u^*, \rho_{2n})\| \\
& + b\|d(\rho_{2n}, u^*) + d(u^*, Pu^*)\| \\
& + c\|d(u^*, Pu^*)\|) \rightarrow 0 \\
& \leq M(\|d(u^*, \rho_{2n+1})\| + a\|d(u^*, \rho_{2n})\| \\
& + b\|d(\rho_{2n}, u^*)\| + b\|d(u^*, Pu^*)\| \\
& + c\|d(u^*, Pu^*)\|) \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Hence $\|d(u^*, Pu^*)\| \leq 0$. Thus $d(u^*, Pu^*) = 0$.

This implies $u^* = Pu^*$. Thus u^* is the fixed point of P .

Similarly, it can be established that, u^* is also fixed point of Q , that is, $u^* = Qu^*$. Hence $Pu^* = u^* = Qu^*$. Therefore, u^* is common fixed point of P and Q .

Now to prove uniqueness: Suppose that v^* is another common fixed point of P and Q , such that $Pv^* = v^* = Qv^*$. Now from (4) we have:

$$\begin{aligned}
& d(u^*, v^*) = d(Pu^*, Qv^*) \\
& \leq a(u^*, v^*) + b \frac{[d(u^*, Qv^*)]^2}{[d(u^*, v^*) + d(v^*, Qv^*)]} \\
& + c \frac{[d(v^*, Qv^*)]^2}{[d(u^*, v^*) + d(u^*, Qv^*)]} \\
& \leq a(u^*, v^*) + b \frac{[d(v^*, v^*)]^2}{[d(u^*, v^*) + d(v^*, v^*)]} \\
& + c \frac{[d(v^*, v^*)]^2}{[d(u^*, v^*) + d(u^*, v^*)]} \\
& \leq (a + b)d(u^*, v^*)
\end{aligned}$$

Hence

$\|d((u^*.v^*))\| \leq M[(a+b)\|d((u^*.v^*))\|] \rightarrow 0$ as $n \rightarrow \infty$.
So, we have $\|d(u^*.v^*)\| = 0$. Implies that $u^* = v^*$ is a unique common fixed point of P and Q . This completes the proof.

Now we will derive and generalize the Theorem 3.1 & Theorem 3.2 for three self-mapping in cone metric space and obtain unique common fixed point.

Theorem 3.5: Let (X, d) be cone metric spaces and let $T, P, Q: X \rightarrow X$ be any three continuous self-mappings on X . Assume that T is one to one, continuous function and P is a normal cone with normal constant. If the mapping T, P and Q satisfy the condition

$$\begin{aligned} d(TP\rho, TQ\sigma) &\leq a(\rho, \sigma) \\ &+ b[d(T\rho, TP\rho) + d(T\sigma, TQ\sigma)] \\ &\times \frac{[d(T\rho, T\sigma) + d(T\sigma, TQ\sigma)]}{[d(T\rho, TQ\sigma)]} \\ &+ c[d(T\rho, TQ\sigma) + d(T\sigma, TP\rho)] \\ &\times \frac{[d(T\rho, T\sigma) + d(T\sigma, TQ\sigma) + d(T\rho, TQ\sigma)]^2}{[d(T\rho, TQ\sigma)]^2} \end{aligned} \quad (6).$$

for all $\rho, \sigma \in X$ where $a, b, c > 0$ With $a + 2b + 8c \leq 1$. then the pairs T, P and Q have a unique common fixed point in X .

Proof: Let $\rho_0 \in X$. We define the iterative sequence $\{\rho_{2j}\}$ and $\{\rho_{2j+1}\}$ by:

$$\rho_{2j+1} = P\rho_{2j} = P^{2j}\rho_0$$

and

$$\rho_{2j+2} = Q\rho_{2j+1} = Q^{2j+1}\rho_0,$$

for each $j = 0, 1, 2, 3 \dots \infty$.

Put $\rho = \rho_{2j-1}$ and $\sigma = \rho_{2j}$ in (6) we have:

$$\begin{aligned} d(T\rho_{2j}, T\rho_{2j+1}) &\leq d(TP\rho_{2j-1}, TQ\rho_{2j}) \\ &\leq ad(T\rho_{2j-1}, T\rho_{2j}) \\ &+ b[T\rho_{2j-1}, TP\rho_{2j-1}) + d(T\rho_{2j}, TQ\rho_{2j})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, TQ\rho_{2j})]}{[d(T\rho_{2j-1}, TQ\rho_{2j})]} \\ &+ c[d(T\rho_{2j-1}, TQ\rho_{2j}) + d(TQ\rho_{2j}, TP\rho_{2j-1})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, TQ\rho_{2j}) + d(T\rho_{2j-1}, TQ\rho_{2j})]^2}{[d(T\rho_{2j-1}, TQ\rho_{2j})]^2} \\ &\leq ad(T\rho_{2j-1}, T\rho_{2j}) \\ &+ b[T\rho_{2j-1}, T\rho_{2n}) + d(T\rho_{2n}, T\rho_{2n+1})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})]}{[d(T\rho_{2j-1}, T\rho_{2j+1})]} \end{aligned}$$

$$\begin{aligned} &+ c[d(T\rho_{2j-1}, T\rho_{2j+1}) + d(T\rho_{2j}, T\rho_{2j})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1}) + d(T\rho_{2j-1}, T\rho_{2j+1})]^2}{[d(T\rho_{2j-1}, T\rho_{2j+1})]^2} \\ &\leq ad(T\rho_{2j-1}, T\rho_{2j}) \\ &+ b[T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})]}{[T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})]} \\ &+ c[T\rho_{2n-1}, T\rho_{2n}) + d(T\rho_{2n}, T\rho_{2n+1})] \\ &\times \frac{[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1}) + d(T\rho_{2j-1}, T\rho_{2j+1})]^2}{[T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})]^2} \\ &\leq ad(T\rho_{2j-1}, T\rho_{2j}) \\ &+ b[T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})] \\ &+ 4c[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})] \\ &\Rightarrow d(T\rho_{2j}, T\rho_{2j+1}) \leq \frac{(a+b+4c)}{(1-b-4c)} d(T\rho_{2j-1}, T\rho_{2j}) \end{aligned}$$

Implies that,

$$d(T\rho_{2j}, T\rho_{2j+1}) \leq \delta d(T\rho_{2j-1}, T\rho_{2j}),$$

$$\delta = \text{where } \frac{(a+b+4c)}{(1-b-4c)} < 1, 0 < \delta < 1.$$

In general, by induction we have

$$d(T\rho_{2j}, T\rho_{2j+1}) \leq \delta d(T\rho_{2j-1}, T\rho_{2j})$$

$$\leq \dots \dots \dots \leq$$

$$\leq \delta^{2n} d(T\rho_0, T\rho_1)$$

So, for $j, k \in N$ with $j < k$ we have

$$\begin{aligned} d(T\rho_{2j}, T\rho_{2k}) &\leq d(\rho_{2n}, \rho_{2n-1}) + d(T\rho_{2j-1}, T\rho_{2j-2}) \\ &+ \dots \dots + d(T\rho_{2j+2k+1}, T\rho_{2k}) \\ &\leq \left(\delta^{2n+1} + \delta^{2n+2} + \dots \right) d(T\rho_0, T\rho_1) \\ &\leq \frac{\delta^{2j}}{1-\delta} d(T\rho_0, T\rho_1) \dots \end{aligned} \quad (7)$$

By definition of normality of cone, so by (7) we get

$$\|d(T\rho_{2j}, T\rho_{2k})\| \leq M \frac{\delta^{2j}}{1-\delta} \|d(T\rho_0, T\rho_1)\|$$

Implies that $d(T\rho_{2j}, T\rho_{2k}) \rightarrow 0$ as $j, k \rightarrow \infty$. Thus $\{T\rho_{2n}\}$ is a Cauchy sequence in X . Since (X, d) is a complete cone metric spaces, there exist $v \in X$ Such that

$$\lim_{n \rightarrow \infty} T\rho_{2n} = v \dots \dots \dots \quad (8)$$

Since T is subsequently convergent, $\{\rho_{2n}\}$ has a convergent subsequence. So, there are $u \in X$ and $\{\rho_{2n(i)}\}$

such that

$$\lim_{i \rightarrow \infty} T\rho_{2n(i)} = u \dots \quad (9)$$

Since T is continuous, then by (3.5.4), we obtain

$$\lim_{i \rightarrow \infty} T\rho_{2n(i)} = Tu \dots, \quad (10)$$

By (8) and (10) we conclude that

$$u = Tu \dots \dots \quad (11)$$

On the other hand

$$\begin{aligned} d(Tu, TPu) &\leq [d(Tu, T\rho_{2j+1}) + d(T\rho_{2j+1}, TPu)] \\ &\leq [d(Tu, T\rho_{2j+1}) + d(T\rho_{2j}, TPu)] \\ &\leq [d(Tu, T\rho_{2j}) + d(T\rho_{2j}, TPu)] \\ &\leq d(Tu, T\rho_{2j}) + ad(T\rho_{2j}, Tu) \\ &\quad + b[(T\rho_{2j} T\rho_{2j}) + d(Tu, TPu)] \\ &\quad \times \frac{[d(T\rho_{2j}, Tu) + d(Tu, TPu)]}{[d(T\rho_{2j}, Tu) + d(Tu, TPu)]} \\ &\quad + c[d(T\rho_{2j}, TPu) + d(Tu, TPu)] \\ &\times \frac{[d(T\rho_{2j}, Tu) + d(Tu, TPu) + d(T\rho_{2j}, Tu) + d(Tu, TPu)]^2}{[d(T\rho_{2j}, Tu) + d(Tu, TPu)]^2} \\ &\leq d(Tu, T\rho_{2j+1}) + ad(T\rho_{2j}, Tu) \\ &\quad + b[d(T\rho_{2j} T\rho_{2j+1}) + d(Tu, TPu)] \\ &\quad + 4c[d(T\rho_{2n}, TPu) + d(Tu, TPu)] \end{aligned}$$

So, using the condition of normality of cone

$$\begin{aligned} \|d(Tu, TPu)\| &\leq M(\|d(Tu, T\rho_{2j+1})\| + a\|d(T\rho_{2n}, Tu)\| \\ &\quad + b\left\| \frac{[d(T\rho_{2n} T\rho_{2j+1})]}{+d(Tu, TPu)} \right\| \\ &\quad + 4c\left\| \frac{[d(T\rho_{2j}, TPu)]}{+d(Tu, TPu)} \right\|) \rightarrow 0 \\ &\leq M(\|d(Tu, T\rho_{2j+1})\| + a\|d(T\rho_{2j}, Tu)\| \\ &\quad + b\|d(T\rho_{2j} T\rho_{2j+1})\| + b\|d(Tu, TPu)\| \\ &\quad + 4c\|d(T\rho_{2j}, TPu)\| + 4c\|d(Tu, TPu)\|) \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

Hence

$$\|d(Tu, TPu)\| \leq 0. \text{ Thus } d(Tu, TPu) = 0.$$

This implies $Tu = TPu$. So, Since T one to one.

Then $Pu = u$ is fixed point of P .

Similarly, it can be established that, u is also fixed point of Q , that is, $u = Qu$. Hence $Pu = u = Qu$. Therefore, u is common fixed point of P and Q .

Now to prove uniqueness: Suppose that v is another common fixed point of P and Q , such that then $Pv = v = Qv$. Now from (6) we have:

$$\begin{aligned} d(Tu, Tv) &= d(TPu, TQv) \\ &\leq a(Tu, Tv) + b[d(Tu, TPu) \\ &\quad + d(Tv, TQv)] \times \frac{[d(Tu, Tv) + d(Tv, TQv)]}{[d(Tu, TQv)]} \\ &\quad + c[d(Tu, TQv) + d(Tv, TPu)] \\ &\quad \times \frac{[d(Tu, Tv) + d(Tv, TQv) + d(Tu, TQv)]^2}{[d(Tu, TQv)]^2} \\ &\leq a(Tu, Tv) + b[d(Tu, Tu) + d(Tv, Tv)] \\ &\quad \times \frac{[d(Tu, Tv) + d(Tv, Tv)]}{[d(Tu, TQv)]} + c[d(Tu, Tv) + d(Tv, Tu)] \\ &\quad \times \frac{[d(Tu, Tv) + d(Tv, Tv) + d(Tu, Tv)]^2}{[d(Tu, Tv)]^2} \\ &\leq a(Tu, Tv) + c[2d(Tu, Tv)] \times \frac{[2d(Tu, Tv)]^2}{[d(Tu, Tv)]^2} \\ &\leq (a + 8c)d(Tu, Tv) \end{aligned}$$

Hence

$$\|d(Tu, Tv)\| \leq M[(a + 8c)\|d(Tu, Tv)\|] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

So, we have:

$\|d(Tu, Tv)\| = 0$. Implies that $Tu = Tv$. Since T is continuous. So, $u = v$ is a unique common fixed point of P and Q . Thus the pair of T, P and Q have unique common fixed point in X . This completes the proof of theorem.

Theorem 3.6: Let (X, d) be cone metric spaces and let $T, P, Q: X \rightarrow X$ be any three continuous self-mappings on X . Assume that T is one to one, continuous function and P is a normal cone with normal constant. If the mapping T, P and Q satisfy the condition:

$$\begin{aligned} d(TP\rho, TQ\sigma) &\leq ad(T\rho, T\sigma) + b \frac{[d(T\rho, TP\sigma)]^2}{[d(T\rho, T\sigma) + d(T\sigma, TQ\sigma)]} \\ &\quad + c \frac{[d(T\sigma, TQ\sigma)]^2}{[d(T\rho, T\sigma) + d(T\rho, TQ\sigma)]} \dots \quad (12) \end{aligned}$$

for all $\rho, \sigma \in X$ where $a, b, c > 0$ With $a + 2b + 4c \leq 1$. then the pairs T, P and Q have a unique common fixed point in X .

Proof: Let $\rho_0 \in X$. We define the iterative sequence $\{\rho_{2j}\}$ and $\{\rho_{2j+1}\}$ by:

$$\rho_{2j+1} = P\rho_{2j} = P^{2j}\rho_0$$

and

$$\rho_{2j+2} = Q\rho_{2j+1} = Q^{2j+1}\rho_0,$$

for each $j = 1, 2, 3 \dots \infty$.

Put $\rho = \rho_{2j-1}$ and $\sigma = \rho_{2j}$ in (12) we have

$$\begin{aligned} d(T\rho_{2j}, T\rho_{2j+1}) &\leq d(TP\rho_{2j-1}, TQ\rho_{2j}) \\ &\leq ad(T\rho_{2j-1}, T\rho_{2j}) \end{aligned}$$

$$\begin{aligned}
& +b \frac{[d(T\rho_{2j-1}, TQ\rho_{2j})]^2}{[d(T\rho_{2j-1}, T\rho_{2n})+d(T\rho_{2j}, TQ\rho_{2j})]} \\
& +c \frac{[d(T\rho_{2j}, TQ\rho_{2j})]^2}{[d(T\rho_{2j-1}, T\rho_{2j})+d(T\rho_{2j-1}, TQ\rho_{2j})]} \\
& \leq ad(T\rho_{2j-1}, T\rho_{2j}) \\
& +b \frac{[d(T\rho_{2j-1}, T\rho_{2j})+d(T\rho_{2j}, T\rho_{2j+1})]^2}{[d(T\rho_{2j-1}, T\rho_{2j})+d(T\rho_{2j}, T\rho_{2j+1})]} \\
& +c \frac{[d(T\rho_{2j-1}, T\rho_{2j})+d(T\rho_{2j-1}, T\rho_{2j+1})]^2}{[d(T\rho_{2j-1}, T\rho_{2j})+d(T\rho_{2j-1}, T\rho_{2j+1})]} \\
& \leq ad(T\rho_{2j-1}, T\rho_{2j}) \\
& +b[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})] \\
& +c[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j-1}, T\rho_{2j+1})] \\
& \leq ad(T\rho_{2j-1}, T\rho_{2j}) \\
& +b[d(T\rho_{2j-1}, T\rho_{2j}) + d(T\rho_{2j}, T\rho_{2j+1})] \\
& +c[d(T\rho_{2j-1}, T\rho_{2j}) \\
& +\{d(T\rho_{2j-1}, T\rho_{2j}) d(T\rho_{2j}, T\rho_{2j+1})\}]
\end{aligned}$$

Therefore,

$$\begin{aligned}
& (1-b-c)d(T\rho_{2j}, T\rho_{2j+1}) \\
& \leq (a+b+c)d(T\rho_{2j-1}, T\rho_{2j}) \\
& d(T\rho_{2j}, T\rho_{2j+1}) \leq \frac{(a+b+2c)}{(1-b-c)} d(T\rho_{2j-1}, T\rho_{2j})
\end{aligned}$$

Implies that,

$$d(T\rho_{2j}, T\rho_{2j+1}) \leq \lambda d(T\rho_{2j-1}, T\rho_{2j}),$$

where $\lambda = \frac{(a+b+2c)}{(1-b-c)} < 1, 0 < \lambda < 1$.

In general, by induction we have

$$\begin{aligned}
d(T\rho_{2j}, T\rho_{2j-1}) & \leq \lambda d(T\rho_{2j-1}, T\rho_{2j}) \\
& \leq \dots \dots \dots \leq \\
& \leq \lambda^{2n} d(T\rho_0, T\rho_1)
\end{aligned}$$

So, for $j, k \in N$ with $j < k$ we have

$$\begin{aligned}
d(T\rho_{2j}, T\rho_{2k}) & \leq d(T\rho_{2j}, T\rho_{2j-1}) + d(T\rho_{2j-1}, T\rho_{2j-2}) \\
& + \dots \dots + d(T\rho_{2j+2k+1}, T\rho_{2k}) \\
& \leq (\lambda^{2n+1} + \lambda^{2n+2} + \dots + \lambda^{2j+2k-1}) \\
& d(T\rho_0, T\rho_1) \leq \frac{\lambda^{2j}}{1-\lambda} d(T\rho_0, T\rho_1) \dots \quad (13)
\end{aligned}$$

Since P is normal cone with normal constant, so by (13) we get

$$\|d(T\rho_{2j}, T\rho_{2k})\| \leq M \frac{\lambda^{2j}}{1-\lambda} \|d(T\rho_0, T\rho_1)\|$$

Implies that $d(T\rho_{2j}, T\rho_{2k}) \rightarrow 0$ as $j, k \rightarrow \infty$. Thus $\{T\rho_{2n}\}$ is a Cauchy sequence, so by completeness of X . This sequence must be convergent in X . Since T is subsequently convergent, so $\{\rho_{2n}\}$ has a convergent subsequence in X at a point $u \in X$.

On the other hand

$$\begin{aligned}
d(Tu, TPu) & \leq [d(Tu, T\rho_{2j+1}) + d(T\rho_{2j+1}, TPu)] \\
& \leq [d(Tu, T\rho_{2j+1}) + d(T\rho_{2j+1}, TPu)] \\
& \leq [d(Tu, TP\rho_{2j}) + d(T\rho_{2j}, TPu)] \\
& \leq d(Tu, TP\rho_{2j}) + ad(Tu, T\rho_{2j}) \\
& +b \frac{[d(T\rho_{2j}, TPu)]^2}{[d(T\rho_{2j}, Tu)+d(Tu, TPu)]} +c \frac{[d(Tu, TPu)]^2}{[d(T\rho_{2j}, Tu)+d(Tu, TPu)]} \\
& \leq d(Tu, T\rho_{2j+1}) + ad(Tu, T\rho_{2j}) \\
& +b \frac{[d(T\rho_{2j}, Tu)+d(Tu, TPu)]^2}{[d(T\rho_{2j}, Tu)+d(Tu, TPu)]} +c \frac{[d(Tu, TPu)]^2}{[d(Tu, TPu)]} \\
& \leq d(Tu, T\rho_{2j+1}) + ad(Tu, T\rho_{2n}) \\
& +b[d(T\rho_{2j}, Tu) + d(Tu, TPu)] \\
& +c d(Tu, TPu)
\end{aligned}$$

Now using the condition of normality of cone

$$\begin{aligned}
\|d(Tu, TRu)\| & \leq M(\|d(Tu, T\rho_{2j+1})\| \\
& +a\|d(Tu, T\rho_{2j})\| \\
& +b\|[d(T\rho_{2j}, Tu) + d(Tu, TPu)]\| \\
& +c\|[d(Tu, TPu)]\|) \rightarrow 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\|d(Tu, TRu)\| & \leq M(\|d(Tu, T\rho_{2j+1})\| + a\|d(Tu, T\rho_{2j})\| \\
& +b\|d(T\rho_{2j}, Tu)\| + b\|d(Tu, TPu)\| \\
& +c\|d(Tu, TPu)\|) \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

Therefore,

$$\|d(Tu, TPu)\| \leq 0. \text{ Thus } d(Tu, TPu) = 0.$$

This implies $Tu = TPu$. So, Since T one to one.

Then $Pu = u$ is fixed point of P .

Similarly, it can be established that, u is also fixed point of Q , that means, u is common fixed point P and Q .

Now to prove uniqueness: Suppose that v^* is another common fixed point of P and Q , such that then $Pv^* = v^* = Qv^*$. Now from (12) we have:

$$d(Tu, Tv) = d(TPu, TQv)$$

$$\begin{aligned}
&\leq a(Tu, Tv) + b \frac{[d(Tu, TQv)]^2}{[d(Tu, Tv) + d(Tv, TQv)]} \\
&\quad + c \frac{[d(Tv, TQv)]^2}{[d(Tu, Tv) + d(Tu, TQv)]} \\
&\leq a(Tu, Tv) + b \frac{[d(Tu, Tv)]^2}{[d(Tu, Tv) + d(Tv, Tv)]} \\
&\quad + c \frac{[d(Tv, TQv)]^2}{[d(Tu, Tv) + d(Tu, TQv)]} \\
&\leq (a + b)d(Tu, Tv)
\end{aligned}$$

Hence

$$\|d(Tu, Tv)\| \leq M[(a + b)\|d(Tu, Tv)\|] \rightarrow 0$$

as $n \rightarrow \infty$. So, we have

$\|d(Tu, Tv)\| = 0$. Implies that $Tu = Tv$. Since T is continuous. So, $u = v$ is a unique common fixed point of R and S . Thus the pair of T, R and S have unique common fixed point in X . This completes the proof.

4. Conclusion and Future Works

In this attempt, we generalize and extend a unique common fixed point result in complete cone metric spaces for two and three self-mapping satisfying rational expression. These results generalize, improve and extend the theorem 3.3 and Theorem 3.4 for two self-mappings and Theorem 3.5 and theorem 3.6 for three self-mapping, which is given by Kumar, P. and Ansari, Z. K. [2].

In future research work, we will look at the problem and derive, expand and generalize in various other cone metric spaces.

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